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Rearrangements and partial differential equations for planar vortices

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Summary

This thesis is concerned with existence of weak solutions of a certain type of a semilinear elliptic partial differential equation in connection with bounded steady vortices in a 2-dimensional ideal fluid. This is done by means of rearrangement variational principles. The variational principles are based on that proposed by Benjamin [6] in which a functional related to the kinetic energy is maximised relative to the set of rearrangements of a prescribed function.

The first three problems have been inspired by Nycander [40]. In Chapter 2 we prove existence of a flow containing a doubly Steiner symmetric and bounded vortex anomaly in a flow approaching a shear flow at infinity. The results of this chapter have been reported in Emamizadeh [21]. The third chapter studies the same problem as in Chapter 1, the difference being that here we use a different variational formulation of the problem. More specifically, we maximise the energy functional relative to the set of rearrangements of a given function which satisfy a prescribed linear constraint. This chapter is a joint work with the author's supervisor, see [13]. The fourth chapter is concerned with the existence of a flow past an impenetrable obstacle containing a bounded vortex anomaly and approaching a shear flow at infinity.

The last three chapters have been motivated by Burton [12]. In Chapter 5 we prove existence of a flow in a quadrant-plane containing a bounded vortex and approaching an irrotational flow at infinity. Here, contrary to [12], the method of Steiner symmetrisation is not available. The vorticity is the maximiser of a functional relative to the set of rearrangements of a prescribed function. The functional is shown to attain a maximum for sufficiently small values of a positive parameter. In Chapter 6 we consider the same problem as in Chapter 5, but use a different variational formulation. Finally, Chapter 7 is concerned with the existence of a flow in a quadrant-plane past an obstacle, attached to the boundary of the domain, containing a bounded vortex approaching an irrotational flow at infinity.

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Chapter 1

Introduction

The idea that for steady, inviscid flows the vorticity should give kinetic energy an extreme value goes back to Kelvin [43], see also Turkington [44]. More recently it has been suggested by Benjamin [6] and the work of Arnold [3] that a natural class in which to seek solutions is the set of rearrangements of a fixed function. Burton [10, 11] has recently developed the required functional analysis for proving existence in a class of rearrangements. The contents of [10, 11] are referred to as *Burton's theory*, in this thesis. Roughly speaking, this theory provides a package for proving certain maximisation problems are solvable, where the set of admissible functions is a rearrangement class generated by a fixed function, on a bounded domain; furthermore for any maximiser *vorticity* is an increasing function of the corresponding *stream function*. Burton's theory relies on maximisation of linear functionals relative to the set of rearrangements of a fixed function. This issue is outlined in due course.

In this thesis we study applications of Burton's theory to six problems. Before outlining our main results we give a general background theory and also survey the relevant results in the literature.

1.1 Background

We begin this section by giving a brief account of basics in rearrangements of functions.

1.1.1 Rearrangements of functions

Definition Let $(\Omega, \mathcal{A}, \mu)$ and $(\Omega', \mathcal{A}', \mu')$ be positive measure spaces with $\mu(\Omega) = \mu'(\Omega')$. Real measurable functions f on Ω and g on Ω' are *rearrangements* of each other if

$$\mu(f^{-1}([\beta, \infty))) = \mu'(g^{-1}([\beta, \infty))), \ \ \forall \beta \in \mathbb{R}, \tag{1.1}$$

provided both sides in (1.1) are finite; if additionally $1 \leq p \leq \infty$ and $f \in L^p(\mu)$ then it follows that $g \in L^p(\mu')$ and $||f||_p = ||g||_p$.

For a non-negative measurable function on Ω, $\mathcal{F}(f_0)$ denotes the set of rearrangements of f_0 on Ω. As a special case Eydeland, Spruck and Turkington [23] proved that if f_0 is defined on the half line $[0, \infty)$ then

$$\mathcal{F}(f_0) = \left\{ f \geq 0 | \int_0^\infty (f - \sigma)^+ = \int_0^\infty (f_0 - \sigma)^+, \ \forall \sigma > 0 \right\},$$

where "+" indicates the positive part of the function.

Ryff [42] showed that for a non-negative function $f_0 \in L^1(0,1)$, the weak closure of the set of rearrangements is convex and therefore equal to the closed convex hull of the set of rearrangements. This result was later generalised by Burton and Ryan [15].

Let us now consider some well known examples of rearrangements. Below, f is assumed to be a non-negative measurable function on $\mathbb{R}^n$ with the property that $\mu_n(f^{-1}([\alpha, \infty))) < \infty$ for all $\alpha > 0$, unless otherwise stated. Here μ_n denotes the $n-$dimensional Lebesgue measure in $\mathbb{R}^n$.

Definition (*Symmetric decreasing rearrangements*)
Let $F(\alpha) := \mu_n(f^{-1}([\alpha, \infty)))$, for $\alpha > 0$. Then the symmetric decreasing rearrangement of f denoted f_Δ is defined by

$$f_\Delta(s) := \begin{cases} \max\{\alpha > 0 | \ F(\alpha) \geq 2|s|\}, & \text{if there is such } \alpha \\ 0, & \text{otherwise,} \end{cases}$$

for $s \in \mathbb{R}$.

Definition(*Steiner-symmetrisation about $x_n = 0$*)
The Steiner symmetrisation of f about the plane $x_n = 0$ denoted f_s is defined by

$$f_s(x_1, \cdots, x_n) := f_\Delta(x', \cdot),$$

where $x' = (x_1, \cdots, x_{n-1})$.
Remark Clearly the definition of Steiner-symmetrisation about any plane can be written similarly.

Definition (*Schwarz-symmetrisation*)

f^* denotes the Schwarz-symmetrisation of f and is defined by

$$f^*(x) := \max\{\alpha|\ F(\alpha) \geq |x|^n\omega_n\},$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. Moreover for any $x_0 \in \mathbb{R}^n$, the *Schwarz-symmetrisation of f with respect to x_0* is simply $f^*(\cdot - x_0)$.

Let us point out that f^* is a spherically symmetric decreasing function. Schwarz-symmetrisations are particularly important in partial differential equations as they, for example, decrease convex gradient integrals. Pólya and Szegö [39] showed that for $1 \leq p < \infty$,

$$\|\nabla f^*\|_p \leq \|\nabla f\|_p, \quad f \in W^{1,p}(\mathbb{R}^n). \tag{1.2}$$

The case of equality in (1.2) has been studied by Brothers and Ziemer [8]. These results have been employed by Burton and McLeod [14] in order to study maximisation and minimisation of the Dirichlet integral of a function vanishing on the boundary of the unit ball, subject to the constraint that the Laplacian be a rearrangement of a given function. They proved that in case the Laplacian is one-signed, maximisers and minimisers are radial and monotone.

Definition Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and $f : \Omega \to \mathbb{R}$ a measurable function. For real α define

$$\lambda_{f,\mu}(\alpha) := \mu(\{x \in \Omega|\ f(x) \geq \alpha\}),$$

called the *distribution function of f with respect to μ*. Then the (essentially) unique *decreasing rearrangement of f*, denoted f^Δ, is defined by

$$f^\Delta(s) := \max\{\alpha|\ \lambda_{f,\mu}(\alpha) \geq s\},$$

for $0 < s < \mu(\Omega)$.

We conclude this section with some rearrangement inequalities.

Theorem 1(Burton[10])
Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, let $1 \leq p \leq \infty$, let q be the conjugate exponent of p and set $\omega = \mu(\Omega)$. Suppose $f_0 \in L^p(\mu)$ and $g_0 \in L^q(\mu)$. Then for all rearrangements f of f_0 and g of g_0 on Ω we have

$$\int_\Omega fg \leq \int_0^\omega f_0^\Delta\, g_0^\Delta.$$

Theorem 2 (Riesz's rearrangement inequality, Lieb [34])
Let f, g and h be three non-negative functions defined on $\mathbb{R}^n$. Then, with

$$I(f,g,h) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(x-y)h(y)dxdy,$$

we have

$$I(f,g,h) \leq I(f^*,g^*,h^*), \tag{1.3}$$

with the understanding that $I(f^*,g^*,h^*)=\infty$ if $I(f,g,h)=\infty$.

Remark The inequality in (1.3) still holds if f and h are non-negative and g is spherically symmetric decreasing.

A generalisation of Theorem 2 is proved by Brascamp, Lieb and Luttinger [7]. The cases of strict inequality and equality in (1.3) have been investigated by Lieb [33] and Burchurd [9], respectively.

1.1.2 Two-dimensional Euler equations

In this section we discuss the fundamental equations describing the motion of a 2-dimensional ideal fluid and establish some elementary facts. The presentation is introductory and informal. Hence, on the one hand we confine ourselves to a very simple flow geometry: the fluid domain $D \subset \mathbb{R}^2$ is bounded and has smooth boundary ∂D and the (steady) flow is everywhere tangential on ∂D. On the other hand, we assume the functions involved are smooth enough to be differentiated as many times as needed.

Let $u(x)=(u_1(x),u_2(x))$ and $p(x)$ denote the velocity field and the pressure of the flow. Then they are required to satisfy the *Euler equations*

$$\begin{aligned}(u\cdot\nabla)u &= -\nabla p, \text{ in } D && (1.4)\\ \nabla\cdot u &= 0, \text{ in } D && (1.5)\\ u\cdot\vec{n} &= 0, \text{ on } \partial D, && (1.6)\end{aligned}$$

where $\vec{n}$ is the exterior unit normal to ∂D.

Remark When the domain is unbounded, the equations of motion (1.4)-(1.6) which have a local character, remain valid. However, in this case, we must specify not only the boundary conditions (1.6) but also the asymptotic behaviour of the velocity field $u(x)$ when $|x|\to\infty$.

We now introduce a fundamental concept of our analysis, namely the *vorticity field* $\vec{\omega}(x)$. By definition

$$\vec{\omega} := \nabla \times u = (0, 0, \partial_{x_1} u_2 - \partial_{x_2} u_1).$$

The *vorticity function* $\omega(x)$ is then defined by

$$\omega(x) := \partial_{x_1} u_2 - \partial_{x_2} u_1.$$

Assuming D is simply connected, the *incompressibility condition* (1.5) allows us to introduce a function ψ, called the *stream function*, such that

$$u = \nabla^{\perp} \psi, \tag{1.7}$$

where $\nabla^{\perp} := (\partial_{x_1}, \partial_{x_2})$. Therefore we derive the well known Poisson equation

$$\Delta \psi = -\omega, \text{ in } D. \tag{1.8}$$

The boundary condition (1.6) implies that $\nabla^{\perp}\psi \cdot \vec{n} = 0$ on ∂D. Assuming $\psi \in C^1(\overline{D})$, it is elementary to show that if $\psi =$ constant on ∂D, then $\nabla^{\perp}\psi \cdot \vec{n} = 0$ on ∂D is satisfied. We assume

$$\psi = 0, \text{ on } \partial D. \tag{1.9}$$

From classical potential theory we deduce that if ω is a given function, then there exists a unique solution for (1.8) satisfying (1.9). Obviously by finding ψ we can find the velocity field from (1.7). therefore we need to write the *dynamical condition* (1.4) in terms of the vorticity function ω. Applying the operator "$\nabla \times$" to (1.4) we obtain, on the account of (1.8) and $\omega = \nabla \times u$,

$$[\psi, \omega] = 0, \text{ in } D, \tag{1.10}$$

where $[\cdot, \cdot]$ denotes the Jacobian. This, in turn, implies that ψ and ω are functionally dependent. In particular, we shall seek solutions for which

$$\omega = \phi \circ \psi, \tag{1.11}$$

which obviously satisfy (1.10). If K denotes the inverse of $-\Delta$ with homogeneous Dirichlet boundary conditions, then from (1.8) we infer

$$\psi = K\omega. \tag{1.12}$$

Inserting (1.12) into (1.11) yields

$$\omega = \phi \circ K\omega. \tag{1.13}$$

We will regard equation (1.13) as fundamental; in proving that a flow is steady, we will consider ourselves to have succeeded when we have checked that (1.13) is satisfied. Hence we need to look for a fixed point of the operator $\phi \circ K$, where ϕ is an unknown function.

Let us point out that in our work ω will always be obtained as a global maximiser of an appropriate *energy* functional relative to the set of rearrangements of a given function, whence (1.13) is the corresponding *Euler-Lagrange equation.* The function ϕ, called the *nonlinearity*, emerges from Burton's theory. The fact that any fixed point of (1.13) gives rise to a solution of Euler's equations (1.4)-(1.6), in some sense, is sketched for a particular situation in [Chapter 7, section 7.6].

1.2 Maximisation of linear functionals

Maximisation of linear functionals forms a cornerstone of Burton's theory. This section is devoted to this issue and some of its applications.

Burton [10, Theorem 4] proves that if $1 \leq p < \infty$, $1/p + 1/q = 1$ and $g \in L^q(D)$, where D is a bounded subset of $\mathbb{R}^2$, then

$$< f, g > := \int_D f(x)g(x)dx$$

attains its supremum relative to $f \in \mathcal{F}(f_0)$, the set of rearrangements of $f_0 \in L^p(D)$. He also proves, [10, Theorem 5], that if $< \cdot, g >$ has a unique maximiser $\hat{f}$ relative to $\mathcal{F}(f_0)$, then there is an increasing function ϕ such that $\hat{f} = \phi \circ g$ almost everywhere in D. To illustrate some applications of these results we consider the problem

$$(P): \sup_{\zeta \in \mathcal{A}} \Psi(\zeta),$$

where $\mathcal{A} \subset L^p(D)$ is the set of admissible functions and $\Psi : L^p(D) \to \mathbb{R}$ is a functional which usually corresponds to some physical phenomenon (e.g. energy). For simplicity of the presentation we focus on a very particular case, namely, we assume that $\mathcal{A} = \mathcal{F}(f_0)$, as above, and that Ψ is convex and weakly sequentially continuous. According to Burton [10, Theorem 6], the weak closure of $\mathcal{F} \equiv \mathcal{F}(f_0)$, denoted $\overline{\mathcal{F}^w}$, in $L^p(D)$ is convex. Ψ, being weakly sequentially continuous, attains a maximum relative to $\overline{\mathcal{F}^w}$, say at ζ_1. Since Ψ is convex and strongly continuous an application of the Hahn-

Banach theorem shows that $\partial\Psi(\zeta_1)$, the subdifferential of Ψ at ζ_1, is non-empty. Let $g \in \partial\Psi(\zeta_1)$; of course we may assume $g \in L^q(D)$. Therefore there exists $\hat{\zeta} \in \mathcal{F}$ that maximises $< \cdot, g >$ relative to $\mathcal{F}$, hence we obtain

$$< \zeta_1, g > \leq < \hat{\zeta}, g > . \tag{1.14}$$

Since g is a subgradient of Ψ at ζ_1 we obtain

$$\Psi(\hat{\zeta}) \geq \Psi(\zeta_1) + < \hat{\zeta} - \zeta_1, g >,$$

hence we conclude, on the account of (1.14), $\Psi(\hat{\zeta}) \geq \Psi(\zeta_1)$. This, in turn, implies that $\Psi(\hat{\zeta}) = \Psi(\zeta_1)$. In other words (P) has a solution. Let us now assume, in addition, that Ψ is strictly convex. Then for $\zeta \in \mathcal{F} \setminus \{\hat{\zeta}\}$ we have

$$\Psi(\hat{\zeta}) \geq \Psi(\zeta) > \Psi(\hat{\zeta}) + < \zeta - \hat{\zeta}, g >,$$

hence $< \zeta - \hat{\zeta}, g >< 0$ This means $\hat{\zeta}$ is the unique maxmiser of $< \cdot, g >$ relative to $\mathcal{F}$. Therefore there is an increasing function ϕ such that

$$\hat{\zeta} = \phi \circ g, \tag{1.15}$$

almost everywhere in D. Assuming that Ψ is Gâteaux differentiable, so $\partial\Psi(\hat{\zeta}) = \{D\Psi[\hat{\zeta}]\}$, from (1.15) we find

$$\hat{\zeta} = \phi \circ D\Psi[\hat{\zeta}]. \tag{1.16}$$

Therefore $\hat{\zeta}$ is a fixed point of $\phi \circ D\Psi[\cdot]$. This, of course, reprises (1.13). So there is a natural question: Is there a link between maximisation of linear functionals and existence of steady vortex flows? The answer is affirmative as we demonstrate below.

Let K be the inverse of $-\Delta$ with homogeneous Dirichlet boundary conditions on D. It is a standard result that for $p \geq 2$, $K : L^p(D) \to H_0^1(D)$ is a compact, symmetric and positive linear operator. Hence $\Psi : L^p(D) \to \mathbb{R}$ defined by

$$\Psi(\zeta) := \frac{1}{2} \int_D \zeta \, K\zeta$$

is weakly sequentially continuous, strictly convex and Gâteaux differentiable with $D\Psi[\zeta] = K\zeta$. Therefore Ψ attains its supremum relative to $\mathcal{F}(f_0)$ and if $\hat{\zeta}$ is a maximiser we have, from (1.16),

$$\hat{\zeta} = \phi \circ K\hat{\zeta},$$

almost everywhere in D. Therefore $\hat{\zeta}$ represents the vorticity of a flow in D. The corresponding stream function is $K\hat{\zeta}$. By global elliptic regularity theory, see e.g. [28, Theorem 8.12], $K\hat{\zeta} \in W^{2,2}(D)$. Hence by the Sobolev embedding theorem $K\hat{\zeta} \in C^{0,\beta}(\overline{D})$ for any $0 < \beta < 1$. Whence $K\hat{\zeta} \in C^{0,\beta}(\overline{D}) \cap H_0^1(D)$. So $K\hat{\zeta}(x) = 0$ for every $x \in \partial D$. This, in turn, implies the flow is tangent at every point on ∂D.

1.3 Survey

In this section we survey the literature related to our work.

The most well known example of steady vortex flow in $\mathbb{R}^2$ is that due to a uniformly translating vortex pair [32]. Fraenkel and Berger [25] proved the existence of a solution of a semilinear elliptic partial differential equation by maximising a certain functional on the surface of a sphere in a Sobolev space. This semilinear equation characterises the steady, unbounded axisymmetric flow of an ideal fluid in $\mathbb{R}^3$ containing bounded regions of vorticity, known as vortex rings. Norbury [38], using analogous methods to those of [25], proved the existence of steady planar vortex pairs in $\mathbb{R}^2$, where the vorticity function, the function ϕ in (1.13), is prescribed and the vortex strength parameter is constrained.

Burton [12] proves the existence of a steady planar flow of an ideal fluid, containing a bounded symmetric pair of vortices, and approaching a uniform flow at infinity. This was done by a rearrangement variational principle in which the data prescribed are the rearrangement class of the vorticity field and either the velocity, $\lambda > 0$, of the vortex pair relative to the fluid at infinity or the momentum impulse, $I > 0$, of the vortex pair. He proves that if λ is sufficiently small or if I is sufficiently large then such flows exist. Since Chapters 5, 6 and 7 are related to this problem we give more detail. When λ is prescribed the corresponding variational formulation of the problem is

$$(P_\lambda): \sup_{\mathcal{F}} \Psi_\lambda(\zeta),$$

where $\mathcal{F}$ is the set of rearrangements of a given function and

$$\Psi_\lambda(\zeta) := \Psi_1(\zeta) - \lambda \Im(\zeta),$$

where

$$\begin{aligned} \Psi_1(\zeta) &:= \frac{1}{2}\int_\Pi \zeta\, T\zeta, \\ \Im(\zeta) &:= \int_\Pi x_2\, \zeta. \end{aligned}$$

Here $\Pi := \{x \in \mathbb{R}^2 | \; x_2 > 0\}$ and T is the inverse of $-\Delta$ with homogeneous boundary conditions on Π. It is proved that (P_λ) is solvable in bounded rectangles and then shown that for λ sufficiently small the maximisers are the same for all sufficiently large rectangles. Hence (P_λ) has a solution. In case I is prescribed a different variational formulation is considered, namely,

$$(\tilde{P}_\lambda): \quad \sup_{\zeta \in \mathcal{F} \cap \Im^{-1}(I)} \Psi_1(\zeta).$$

Turkington [44, 45] uses maximisation of kinetic energy over a different set of functions to prove existence of vortex pairs in flows occupying the whole of $\mathbb{R}^2$ or $D := \mathbb{R}^2 \backslash \tilde{D}$ where $\tilde{D}$ is a bounded simply connected domain, symmetric in the x_1-axis, containing the origin in its interior and having smooth boundary. In case of flows past an obstacle (i.e. $D := \mathbb{R}^2 \setminus \tilde{D}$) he considers the following maximisation problem

$$(T_Q): \quad \sup_{\omega \in K_\lambda(D)} E_Q(\omega),$$

where

$$E_Q(\omega) := \frac{1}{2} \int_D \omega \, G\omega - Q \int_D \eta \, \omega,$$

with G being the inverse of $-\Delta$ with homogeneous Dirichlet boundary conditions on D and

$$K_\lambda(D) := \{\omega \in L^\infty(D) | \; \|\omega\|_1 = 1, \; 0 \leq \omega(x) \leq \lambda \text{ a.e. in } D\}.$$

It is shown that if $a = (a_1, a_2)$ is fixed such that $a_1, \; a_2 \geq 2\max\{1, AQ^{-1}\}$, A is some constant depending on the points where an appropriate Routh function attains its minimum, then the support of any maximiser $\omega := \omega^a_{Q,\lambda}$ of E_Q relative to $K^a_\lambda(D)$, a subset of $K_\lambda(D)$ comprising functions vanishing outside the rectangle $[-a_1, a_1] \times [0, a_2]$, is contained in $[1 - a_1, a_1 - 1] \times [0, a_2 - 1]$, provided λ is sufficiently large. This, of course, implies that (T_Q) is solvable for sufficiently large λ.

Elcrat and Miller [19] use the ideas of Burton [10, 11] and Turkington [44, 45] to prove the existence of a steady planar vortex flow past an obstacle (not necessarily symmetric), approaching a uniform flow at infinity and satisfying a prescribed circulation around the boundary of the obstacle, the domain of the fluid D being unbounded. They do this by maximising the energy functional relative to the set of rearrangements of a given function ζ_λ, such that the measure of the support of ζ_λ is $O(1/\lambda)$, which vanish outside a compact subset of D containing a local minimum of the corresponding Routh function.

Later, Elcrat and Miller [20] used same ideas as in [19], see also [18], and proved the existence of a flow past a multiple number of obstacles in $\mathbb{R}^2$ with prescribed circulations around the boundary of each, approaching a uniform shear flow at infinity with vorticity concentrated around each of the critical points of the corresponding Routh function, say $x_0^1, \cdots, x_0^n$. The vorticity around each critical point x_0^i belongs to the set of rearrangements of a given function, say f_i, vanishing outside a compact set $R_i \subset D$ containing x_0^i.

Badiani [4] also used the methods of Burton [12] and Turkington [44, 45] to prove an existence theorem for a steady planar flow past a symmetric obstacle, containing a symmetric vortex pairs and approaching a uniform flow at infinity.

Coherent vortices are often observed in natural shear flows. Their vorticity usually has the same sign as that of the background flow, and they are elongated in the direction of the flow. Nycander [40] proved the existence of a steady two dimensional flow, in $\mathbb{R}^2$, which is separated into two regions of positive constant and of positive non-constant vorticity, where the former is unbounded whereas the latter is bounded. This was done by maximising an appropriate energy functional relative to the class of rearrangements of a fixed function ζ_0, $\mathcal{F}(\zeta_0)$, vanishing outside bounded sets and satisfying

$$0 < \zeta_{\min} \leq \zeta \leq \zeta_{\max} < \infty, \ \ \zeta \in \mathcal{F}(\zeta_0), \tag{1.17}$$

where $\zeta_{\min}$ and $\zeta_{\max}$ are constants. Because of the nature of the functional the set of admissible functions was reduced to just those in $\mathcal{F}(\zeta_0)$ which are doubly Steiner-symmetric about the axes. His proof is also based on the method suggested by Benjamin [6]. However, the condition that the vorticity (anomaly) has positive lower bound $\zeta \geq \zeta_{\min}$ is quite unreasonable since it may lead to the discontinuity of the vorticity (anomaly) on the boundary of its support (the free boundary) and then create complex phenomenon such as shock waves, etc.

Recently we have learned of a Physics preprint by Wu and Mu [46] concerning the removal of condition (1.17).

1.4 Outline of main results

In Chapter 2 we extend earlier work of Nycander [40]. We prove the existence of a planar uniform shear flow, in $\mathbb{R}^2$, containig a bounded vortex anomaly. The main feature of our work is the removal of condition (1.17), see also Emamizadeh [21].

Chapter 3 deals with the same problem, physically speaking, as in Chapter 2. Here we employ a different variational formulation. More precisely we maximise a non-convex functional relative to a rearrangement class of functions satisfying an equality constraint. This Chapter is a joint work with G.R. Burton, see Burton and Emamizadeh

[13].

Chapter 4 is concerned with proving existence of a flow past an obstacle, containing a bounded vortex anomaly, approaching a shear flow at infinity and satisfying a prescribed circulation around the boundary of the obstacle.

Remark Chapters 2, 3 and 4 have all been motivated by Nycander [40].

In Chapter 5 we prove the existence of steady ideal fluid flows occupying Π_+, the first quadrant) containing a bounded vortex. Such a flow will be described by a stream function $\psi : \Pi_+ \to \mathbb{R}$. At infinity we will have $\psi \to -\lambda x_1 x_2$ which is the stream function for an irrotational flow with velocity field $-\lambda(x_1, -x_2)$, $\lambda > 0$ prescribed. We prove existence of such flows for sufficently small λ.

Chapter 6 addresses the same problem, physically speaking, as in Chapter 5, but here we use a different variational formulation. We assume the impulse, I, given by

$$I = \int_{\Pi_+} x_1 x_2 \, \zeta$$

is prescribed.

In Chapter 7 we use ideas of Burton [12] and Turkington [44, 45] to prove existence of a flow past an obstacle and approaching an irrotational flow at infinity. The domain of the fluid is $\Pi_+ \setminus D$ with $\partial(\Pi_+ \setminus D)$ smooth. The corresponding problem in a half-plane with an obstacle was considered by Badiani [4], and we make extensive use of his methods in our work.

Chapter 2

Steady vortex in a uniform shear flow of an ideal fluid

2.1 Introduction

In this chapter we extend earlier work of Nycander [40]; in which isolated vortices in a background flow of constant shear were studied; here a flow in $\mathbb{R}^2$ is called *shear* if the fluid particles in the upper half plane move in the negative x_1-direction with velocity increasing along the x_2-direction, similarly particles in the lower half plane move in the positive x_1-direction with velocities increasing along the negative x_2-direction. In [40] the author proves existence of a steady two dimensional flow, in $\mathbb{R}^2$, which is separated into two regions of positive constant and of positive non-constant vorticity, in this case the vortex is called *vortex anomaly*, where the former is unbounded and the latter is bounded. This was done by maximising the energy functional relative to a class of rearrangements of a given function ζ_0 that vanishes outside a bounded set and satisfies

$$0 < \zeta_{min} \leq \zeta_0 \leq \zeta_{max} < \infty, \tag{2.1}$$

where ζ_{min} and ζ_{max} are constants. Later in the paper [40] the author states a conjecture to which an affirmative answer would make it possible to remove (2.1). However, in this chapter we present a direct method to eliminate (2.1).

The mathematical difficulty is the lack of compactness which is caused by the unboundedness of the domain of interest, namely $\mathbb{R}^2$. To overcome this difficulty we follow a method proposed by Benjamin [6]; first we consider the maximisation problem in a bounded set and, using Burton's theory [10, 11], prove existence of maximisers. Then we show that for sufficiently large bounded domains the maximisers are the same.

2.2 Notation, definitions and statement of the main result

Henceforth p is a real number in $(2,\infty)$. x, y, z, etc. stand for points in $\mathbb{R}^2$, hence $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2)$, etc. $D(l)$ will always represent a square of the form $[-l,l] \times [-l,l]$. When there is no confusion we simply write D in place of $D(l)$. In particular, we denote by D_0 the square with $l := l_0 = 2^{-1}\sqrt{\pi}a$, for some fixed $a > 0$. $B_r(x)$ stands for the open disc centered at x with radius r. B_r will denote the ball centered at the origin with radius r; particularly note that $D_0 \supset B_a$. For a measurable set $A \subseteq \mathbb{R}^2$, $|A|$ denotes the 2-dimensional Lebesgue measure of A. If A is a measurable set in $\mathbb{R}^2$ and $x \in \mathbb{R}^2$, we say A is dense at x if and only if $|A \cap B_\epsilon(x)| \neq 0$ for all $\epsilon > 0$. The set consisting of the points where A is dense at is denoted $\mathrm{den}(A)$.

Let $\zeta_0 \in L^p(\mathbb{R}^2)$ be a non-negative function which vanishes outside a set of measure πa^2. The set of all rearrangements of ζ_0 on $\mathbb{R}^2$ which vanish outside bounded sets is denoted by $\mathcal{F}$. The subset of $\mathcal{F}$ comprising functions vanishing outside the square D is denoted $\mathcal{F}(D)$; henceforth we assume $D \supseteq D_0$ in order to ensure $\mathcal{F}(D) \neq \emptyset$. Let $S > 0$; then for a non-negative $\zeta \in L^p(\mathbb{R}^2)$ having bounded support, we define the energy functional

$$\Psi(\zeta) := \frac{1}{2}\int_{\mathbb{R}^2} \zeta K\zeta - \frac{S}{2}\int_{\mathbb{R}^2} x_2^2 \zeta,$$

where

$$K\zeta(x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{1}{|x-y|}\zeta(y)dy.$$

P will denote the following maximisation problem

$$P: \sup_{\zeta\in\mathcal{F}} \Psi(\zeta),$$

and the set of solutions of P is denoted Σ. If the maximisation is performed on a square D then we designate it $P(D)$; that is,

$$P(D): \sup_{\zeta\in\mathcal{F}(D)} \Psi(\zeta).$$

Similarly the corresponding set of solutions is denoted $\Sigma(D)$. In [40], the main result is the following

Theorem 1 *Let ζ_0 vanishes outside a bounded set in $\mathbb{R}^2$ and satisfy (2.1). Then P has a solution; that is, $\Sigma \neq \emptyset$.*

The author of [40] has stated a conjecture to which an affirmative answer would help to remove condition (2.1). However, we present a much simpler method to eliminate (2.1) and indeed prove

Theorem 2 *Problem P has a solution; and if $\zeta \in \Sigma$, then $\psi_1 := K\zeta$ will satisfy the following semi-linear elliptic partial differential equation*

$$-\Delta\psi_1 = \phi \circ (\psi_1 - 2^{-1}Sx_2^2), \ a.e. \ in \, \mathbb{R}^2, \tag{2.2}$$

where ϕ is an increasing function unknown a priori.

During the course of this chapter we make use of several types of rearrangements, namely, Steiner, Schwarz and decreasing rearrangements. For the sake of completeness we give the corresponding definitions in the context we need.

A measurable function $h : \mathbb{R}^2 \to \mathbb{R}$ will be called Steiner-symmetric with respect to the line $x_1 = 0$ if

$$x_2 > 0, 0 \leq x_1 \leq x_1' \Rightarrow h(-x_1, x_2) = h(x_1, x_2) \geq h(x_1', x_2), \ \text{ a.e. } (x_1, x_2) \in \mathbb{R}^2.$$

Similarly, one can define Steiner-symmetry with respect to the line $x_2 = 0$. A function which is Steiner-symmetric with respect to both lines $x_1 = 0$ and $x_2 = 0$ is said to be doubly Steiner-symmetric, denoted (DSS). If $h : \mathbb{R}^2 \to \mathbb{R}$ is non-negative, measurable and vanishes outside a set of finite measure, then there is an essentially unique rearrangement $h^\sharp$ of h, called the Steiner-symmetrisation of h, with respect to the line $x_1 = 0$, such that $h^\sharp$ is Steiner-symmetric about the line $x_1 = 0$ and for almost every $x_2 > 0$ and every $\alpha > 0$ the sets where $h(\cdot, x_2) \geq \alpha$ and $h^\sharp(\cdot, x_2) \geq \alpha$ have equal 1-dimensional Lebesgue measure. Similarly, the Steiner-symmetrisation of h with respect to the line $x_2 = 0$ is defined and designated by $h_\sharp$.

The Schwarz-symmetrisation of h is denoted by h^* and satisfies the following conditions:

(i) h^* is a function of $|x|$ and is decreasing.

(ii) $|\{x \in \mathbb{R}^2 | \, h^*(x) \ \geq \ s\}| = |\{x \in \mathbb{R}^2 | \, h(x) \ \geq \ s\}|, \ \forall s > 0.$

If f is a non-negative, measurable function in $\mathbb{R}^2$, then the following inequalities are special cases of Riesz's inequality, see [31]

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x) \log \frac{1}{|x-y|} f(y) dx dy \ \leq \ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^*(x) \log \frac{1}{|x-y|} f^*(y) dx dy \tag{2.3}$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x) \log \frac{1}{|x-y|} f(y) dx dy \ \leq \ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^\sharp(x) \log \frac{1}{|x-y|} f^\sharp(y) dx dy \tag{2.4}$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x) \log \frac{1}{|x-y|} f(y) dx dy \ \leq \ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\sharp(x) \log \frac{1}{|x-y|} f_\sharp(y) dx dy. \tag{2.5}$$

Suppose $h : \mathbb{R}^2 \to \mathbb{R}$ is a non-negative measurable function vanishing outside a bounded set of measure m. Then h has a decreasing rearrangement, $h^{\triangle}$, defined on the interval $(0,m)$ which is a decreasing function satisfying

$$|\{\xi \in (0,m) |\, h^{\triangle}(\xi) \geq s\}| = |\{x \in \mathbb{R}^2 |\, h(x) \geq s\}|, \;\; \forall s > 0.$$

Then $h^{\triangle}$ is uniquely defined except for the values at its discontinuities. If $f \in L^p(\mathbb{R}^2)$ and $g \in L^q(\mathbb{R}^2)$, where q is the conjugate exponent of p, such that f and g vanish outside a set of measure ω then the inequalities

$$\int_{\mathbb{R}^2} fg \leq \int_0^{\omega} f^{\triangle} g^{\triangle}, \tag{2.6}$$

and

$$\int_{\mathbb{R}^2} fg \leq \int_{\mathbb{R}^2} f^*\, g^* \tag{2.7}$$

are classical, see [31]. The reader is refered to [34] for a comprehensive treatment of integral inequalities involving rearrangements of functions.

For a measurable A in $\mathbb{R}^2$, χ_A will denote the characteristic function of A; that is

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A. \end{cases}$$

If $x \in \mathbb{R}^2$, then $d(x, A)$ stands for the distance from x to $\mathrm{den}(A)$. $\mathrm{diam}(A)$ denotes the diameter of $\mathrm{den}(A)$ which is defined by

$$\mathrm{diam}(A) := \sup\{|x - y| : \; x, \; y \in \mathrm{den}(A)\}.$$

For $h : \mathbb{R}^2 \to \mathbb{R}$, measurable, the strong support of h is the set $\{x |\, h(x) > 0\}$ and it is denoted $\mathrm{supp}(h)$. Finally, for real valued measurable functions f and g defined on $\mathbb{R}^2$ we set $< f, g >:= \int_{\mathbb{R}^2} f\, g$, whenever the integral exists.

2.3 Preliminary results

In this section we present properties of the operator K, which are relevant to our work, and some results from Burton's theory.

2.3.1 Properties of K

Our first result concerns the regularity of $K\zeta$, where ζ is a function in $L^p(\mathbb{R}^2)$ vanishing outside a bounded set. In the literature $K\zeta$ is called the Newtonian potential with density ζ; the reader is referred to [24, 28].

Lemma 1 *Let $\zeta \in L^p(\mathbb{R}^2)$ vanish outside a bounded set. Then $K\zeta \in C^1(\mathbb{R}^2)$. Moreover, the following holds*

$$\nabla K\zeta(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\zeta(y)dy, \; \forall x \in \mathbb{R}^2.$$

Proof. See [3, Appendix A].$\diamondsuit$

Remark Suppose ζ is a non-negative function in $L^p(\mathbb{R}^2)$, vanishing outside a bounded set with $|\mathrm{supp}(\zeta)| = \pi a^2$, then by applying a classical result of Hardy, see for example [34], we obtain

$$K\zeta(x) \leq \frac{1}{2\pi}\int_{B_a(x)} \log\frac{1}{|x-y|}\eta(y)dy, \; \forall x \in \mathbb{R}^2,$$

where η is the Schwarz-symmetrisation of ζ with respect to x; that is, $\eta(y) = \zeta^*(y-x)$. Hence, an application of the Hölder's inequality yields

$$K\zeta(x) \leq \frac{1}{2\pi}\left(\int_{B_a(x)} \left|\log\frac{1}{|x-y|}\right|^q dy\right)^{1/q} ||\zeta||_p, \; \forall x \in \mathbb{R}^2.$$

Elementary calculations show that

$$\int_{B_a(x)} |\log\frac{1}{|x-y|}|^q dy \leq C\Gamma(a+1) + O(a(\log a)^2),$$

as $a \to \infty$, where $\Gamma(\cdot)$ stands for the Gamma function. Therefore

$$K\zeta(x) \leq C||\zeta||_p, \; \forall x \in \mathbb{R}^2, \tag{2.8}$$

where C is a constant depending on a. Moreover, if $s = d(x, \mathrm{supp}(\zeta)) + \mathrm{diam}(\mathrm{supp}(\zeta))$, then similar calculations yield

$$|K\zeta(x)| \leq C||\zeta||_p, \; \forall x \in \mathbb{R}^2, \tag{2.9}$$

where C depends only on s.

Lemma 2 *Let $q \geq 1$ and let U be a bounded open subset of $\mathbb{R}^2$. Then $K : L^p(U) \to L^q(U)$ is compact, in the sense that if $\{\zeta_n\}_{n=1}^{\infty}$ is a sequence of functions , bounded in $L^p(\mathbb{R}^2)$ and vanishing outside U, then the restrictions to U of the $K\zeta_n$'s have a subsequence converging in the L^q-norm. Moreover, if $\zeta \in L^p(\mathbb{R}^2)$ and vanishes outside U then $u = K\zeta$ verifies the following Poisson's equation*

$$-\Delta u = \zeta, \; a.e. \; in \, \mathbb{R}^2. \tag{2.10}$$

Proof. From Lemma 1 it follows that the map K from $L^p(U)$ into $L^q(U)$ is well defined. Now consider $\zeta \in L^p(\mathbb{R}^2)$ which vanishes outside U. Then there exists a sequence $\{\zeta_n\}_{n=1}^{\infty}$ in $C_0^\infty(\mathbb{R}^2)$ such that $\mathrm{supp}(\zeta_n) \subset U$, for all $n \in \mathbb{N}$, and $\zeta_n \to \zeta$, in $L^p(\mathbb{R}^2)$, as $n \to \infty$. For $x \in \mathbb{R}^2$, an application of (2.9) implies that

$$|K(\zeta_n - \zeta)(x)| \leq C\|\zeta_n - \zeta\|_p.$$

Therefore, $K\zeta_n \to K\zeta$, as $n \to \infty$, uniformly on compact subsets of $\mathbb{R}^2$. Whence

$$< K\zeta_n, \phi > \to < K\zeta, \phi >, \text{ as } n \to \infty, \ \forall \phi \in C_0^\infty(\mathbb{R}^2). \tag{2.11}$$

Note that $-\Delta K\zeta_n = \zeta_n$, for all $n \in \mathbb{N}$, see for example [28, Lemmas 4.1 and 4.2]. Thus

$$< -\Delta K\zeta_n, \phi > = < \zeta_n, \phi >, \ \forall n \in \mathbb{N}.$$

Hence by an application of the Lebesgue dominated convergence theorem, we have

$$< -\Delta K\zeta_n, \phi > \to < \zeta, \phi >, \text{ as } n \to \infty.$$

We now apply integration by parts to obtain

$$< -\Delta K\zeta_n, \phi > = < K\zeta_n, -\Delta\phi >, \ \forall n \in \mathbb{N}.$$

From (2.11) and the Lebesgue dominated convergence theorem we deduce

$$< -\Delta K\zeta_n, \phi > \to < K\zeta, -\Delta\phi >, \text{ as } n \to \infty.$$

Hence $< \zeta, \phi > = < K\zeta, -\Delta\phi >$, for all $\phi \in C_0^\infty(\mathbb{R}^2)$, which implies

$$-\Delta K\zeta = \zeta \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^2). \tag{2.12}$$

Now by Agmon's regularity theory [2, Theorem 6.1], applied to (2.12), we infer that $K\zeta \in W^{2,p}_{loc}(\mathbb{R}^2)$. Therefore, equation (2.12) holds almost everywhere in $\mathbb{R}^2$. In view of the compact embedding, $W^{1,2}(U) \hookrightarrow\hookrightarrow L^q(U)$, for all $q \geq 1$, see [1], in order to show compactness of K it suffices to prove the boundedness of K as a map from $L^p(U)$ into $W^{1,2}(U)$. To do this, we claim

$$|\nabla K\zeta(x)| \leq C\|\zeta\|_p, \ \forall x \in \mathbb{R}^2, \tag{2.13}$$

where C is constant independent of x. To prove (2.13), we apply Lemma 1. Indeed we have

$$|\nabla K\zeta(x)| \leq \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{|\zeta(y)|}{|x-y|}dy, \ \forall x \in \mathbb{R}^2.$$

Hence if η denotes the Schwarz-symmetrisation of $|\zeta|$ with respect to x, then a classical inequality of Hardy, see [34], yields

$$|\nabla K\zeta(x)| \leq \frac{1}{2\pi}\int_{B_\alpha(x)} \frac{1}{|x-y|}\eta(y)dy,$$

where α satisfies $\pi\alpha^2 = |supp(|\zeta|)|$. Hence, from Hölder's inequality and the fact that $q < 2$ we obtain (2.13). This, in turn, implies that

$$||\nabla K\zeta||_{2,U} \leq C||\zeta||_p|U|^{\frac{1}{2}}.$$

Also, from (2.9), we have

$$||K\zeta||_{2,U} \leq C||\zeta||_p|U|^{\frac{1}{2}}.$$

Therefore, $||K\zeta||_{W^{1,2}(U)} \leq C||\zeta||_p$, hence K is bounded, as desired. This comletes the proof of the lemma.$\diamondsuit$

The following corollary is an immediate consequence of Lemma 2.

Corollary If U is as in Lemma 2, then the energy functional Ψ is weakly sequentially continuous on $L^p(U)$.

2.3.2 Two results from Burton's theory

The following two lemmas are proved in [10] and [11], respectively.

Lemma 3 *If D is a square, then*

(i) $\overline{\mathcal{F}(D)}^w = \bar{co}\ \mathcal{F}(D)$.

(ii) $\overline{\mathcal{F}(D)}^w$ *is weakly sequentially compact.*

Where $\overline{\mathcal{F}(D)}^w$, $\bar{co}\ \mathcal{F}(D)$ denote the weak closure and the closed convex hull of $\mathcal{F}(D)$ in $L^p(D)$, respectively.

Lemma 3 will help us to prove $P(D)$ has a solution.

Lemma 4 *Let O be an open, bounded set in $\mathbb{R}^2$. Let*

$$\mathcal{L}u := \sum_{1\leq|\alpha|\leq m} \mathcal{A}^\alpha(x)\, \mathcal{D}^\alpha u$$

define a linear partial differential operator in O, where $\mathcal{A}^\alpha$ are measurable, finite almost everywhere, and there is no 0-th order term. Let $2 < p < \infty$ and p^* be the conjugate exponent of p. Let $\zeta_0 \in L^p(O)$ be non-negative and let $\mathcal{F}(O)$ be the set of rearrangements of ζ_0 on O and suppose $g \in L^{p^*}(O) \cap W^{m,1}_{loc}(O)$. Suppose $\hat{\zeta} \in \overline{\mathcal{F}(O)^w}$ maximises $< \cdot, g >$ relative to $\overline{\mathcal{F}(O)^w}$ and that $\mathcal{L}g \geq \hat{\zeta}$ almost everywhere in O. Then $\hat{\zeta} \in \mathcal{F}(O)$ and there is an increasing function ϕ such that $\hat{\zeta} = \phi \circ g$ almost everywhere in O.

2.4 Some auxiliary lemmas

Lemma 5 *Let $D \supseteq D_0$. Then $P(D)$ has a solution. Moreover, if $\hat{\zeta}$ is a solution of $P(D)$, then*

$$\hat{\zeta} = \phi \circ (K\hat{\zeta} - 2^{-1}Sx_2^2), \text{ a.e. in } D,$$

where ϕ is an increasing function, unknown a priori.

Proof. Since $\overline{\mathcal{F}(D)^w}$ is weakly sequentially compact and Ψ is weakly sequentially continuous on $L^p(D)$ we infer that Ψ will attain a maximum relative to $\overline{\mathcal{F}(D)^w}$. Let ζ' be a maximiser of Ψ relative to $\overline{\mathcal{F}(D)^w}$, and consider $\zeta \in \overline{\mathcal{F}(D)^w}$. For $t \in [0,1]$, $\zeta' + t(\zeta - \zeta') \in \overline{\mathcal{F}(D)^w}$, since $\overline{\mathcal{F}(D)^w}$ is convex, by Lemma 3. Hence, by calculating the first variation of Ψ at ζ' we obtain

$$\Psi(\zeta' + t(\zeta - \zeta')) = \Psi(\zeta') + tD\Psi[\zeta'](\zeta - \zeta') + o(t), \text{ as } t \to 0^+,$$

where $D\Psi$ denotes the Gâteaux-derivative of Ψ. This, in turn, along with the fact that ζ' is a maximiser yields $D\Psi[\zeta'](\zeta - \zeta') \leq 0$. Thus ζ' maximises $D\Psi[\zeta'](\cdot)$ relative to $\overline{\mathcal{F}(D)^w}$. Note that we may identify $D\Psi[\zeta']$ with $K\zeta' - 2^{-1}Sx_2^2$, an element of $L^q(D)$, where q is the conjugate exponent of p. Hence, ζ' maximises $< \cdot, K\zeta' - 2^{-1}Sx_2^2 >$ relative to $\overline{\mathcal{F}(D)^w}$. We claim that the level sets of $K\zeta' - 2^{-1}Sx_2^2$ have measure zero. To prove the claim assume the contrary, then for some γ the set $L_\gamma := \{x |\ K\zeta' - 2^{-1}Sx_2^2 = \gamma\}$ has positive measure. Since, by Lemma 2, $K\zeta' - 2^{-1}Sx_2^2 \in W^{2,p}_{loc}(\mathbb{R}^2)$, so $K\zeta' - 2^{-1}Sx_2^2$ being constant on L_γ we may apply [28, Lemma 7.1] to deduce that $-\Delta(K\zeta' - \frac{S}{2}x_2^2) = 0$, for almost every $x \in L_\gamma$. However, this is impossible since, by Lemma 2, $-\Delta(K\zeta' - 2^{-1}Sx_2^2) = \zeta' + S > 0$, for almost every $x \in \mathbb{R}^2$. Now that the level sets of $K\zeta' - 2^{-1}Sx_2^2$ have measure zero we can apply Lemma 4 to deduce $\zeta' \in \mathcal{F}(D)$ and that there exists an increasing function, say ϕ', such that

$$\zeta' = \phi' \circ (K\zeta' - 2^{-1}Sx_2^2), \text{ a.e. in } D.$$

Now if $\hat{\zeta} \in \Sigma(D)$, then once again calculating the first variation of Ψ at $\hat{\zeta}$ and proceeding exactly as above we will find an increasing function ϕ such that $\hat{\zeta} = \phi \circ (K\hat{\zeta} - \frac{S}{2}x_2^2)$, for almost every $x \in D$. This completes the proof. $\diamondsuit$

Let $\mathcal{F}_{dss}(D)$ denote the set consisting of functions in $\mathcal{F}(D)$ which are DSS; accordingly $\Sigma_{dss}(D)$ denotes the set of functions in $\Sigma(D)$ which are DSS.

Lemma 6 *Let $D \supseteq D_0$. Then $\Sigma_{dss}(D) \neq \emptyset$.*

Proof. From Lemma 5, there exists $\hat{\zeta} \in \Sigma(D)$. Let us now decompose Ψ as follows

$$\Psi(\zeta) = \Psi_1(\zeta) + \Im(\zeta),$$

where

$$\begin{aligned}\Psi_1(\zeta) &:= \frac{1}{2}\int_{\mathbb{R}^2} \zeta K\zeta, \\ \Im(\zeta) &:= -\frac{S}{2}\int_{\mathbb{R}^2} x_2^2\zeta.\end{aligned}$$

From (2.4) and (2.5), we deduce that $\Psi_1(\hat{\zeta}) \leq \Psi_1(\hat{\zeta}^{\sharp})$ and $\Psi_1(\hat{\zeta}) \leq \Psi_1(\hat{\zeta}_{\sharp})$, respectively. Furthermore, we have $\Im(\hat{\zeta}) = \Im(\hat{\zeta}^{\sharp})$ and $\Im(\hat{\zeta}) \leq \Im(\hat{\zeta}_{\sharp})$, where the second relation follows from a classical inequality, by (2.7). Therefore, we infer

$$\begin{aligned}\Psi(\hat{\zeta}) &\leq \Psi(\hat{\zeta}^{\sharp}), \\ \Psi(\hat{\zeta}) &\leq \Psi(\hat{\zeta}_{\sharp}).\end{aligned}$$

It now follows that $(\hat{\zeta}^{\sharp})_{\sharp}$ is a maximiser of Ψ relative to $\mathcal{F}(D)$, hence $\Sigma_{dss}(D) \neq \emptyset$. This completes the proof of the lemma. $\diamondsuit$

Lemma 7 *Let $D \supseteq D_0$ and $\zeta \in \Sigma(D)$. If $x_0 := (x_{0,1}, x_{0,2}) \in den(supp(\zeta))$, then*

$$K\zeta(x_0) - \frac{S}{2}x_{0,2}^2 > K\zeta(x) - \frac{S}{2}x_2^2\ ,\ a.e.\ in\ D \setminus supp(\zeta).$$

Proof. Applying Lemma 5, we obtain

$$\zeta = \phi \circ (K\zeta - \frac{S}{2}x_2^2),\ \text{a.e. in } D,$$

where ϕ is an increasing function. If we set $\psi(x) := K\zeta(x) - \frac{S}{2}x_2^2$, then $\zeta = \phi \circ \psi$, almost everywhere in D. We now define the set

$$M := \{z \in D|\ \zeta(z) = \phi \circ \psi(z) \text{ and } \psi(z) > \psi(x_0)\}.$$

Let $z_0 \in M$. Then $\psi(z_0) > \psi(x_0) + \epsilon$, for some $\epsilon > 0$. Now, applying the continuity of ψ at x_0, we can find $\delta > 0$ such that $|\psi(x) - \psi(x_0)| < \epsilon$, provided $x \in B_\delta(x_0) \cap D$. Therefore, $\psi(x) < \psi(x_0) + \epsilon < \psi(z_0)$, provided $x \in B_\delta(x_0) \cap D$. If we set $U := B_\delta(x_0) \cap \text{supp}(\zeta)$, then U will have positive measure, since $x_0 \in \text{den}(\text{supp}(\zeta))$. Whence, $\psi(x) < \psi(z_0)$, for all $x \in U$. Invoking the monotonicity of ϕ, we deduce that $\phi \circ \psi(z_0) \geq \phi \circ \psi(x)$, for all $x \in U$. Since $z_0 \in M$, it follows that $\zeta(z_0) \geq \phi \circ \psi(x)$, for all $x \in U$. This, in turn, implies that $\zeta(x) \leq \zeta(z_0)$, for almost every $x \in U$. Since $\zeta(x) > 0$, for all $x \in \text{supp}(\zeta)$, we infer that $\zeta(z_0) > 0$; that is $z_0 \in \text{supp}(\zeta)$. z_0 being an arbitrary element of M, we obtain $M \subseteq \text{supp}(\zeta)$, or $D \setminus \text{supp}(\zeta) \subseteq D \setminus M$. Therefore, $\psi(x) \leq \psi(x_0)$, for almost every $x \in D \setminus \text{supp}(\zeta)$. On the other hand, since the level sets of ψ have measure zero, see the proof of Lemma 5, we obtain $\psi(x) < \psi(x_0)$, for almost every $x \in D \setminus \text{supp}(\zeta)$. This completes the proof of the lemma. $\diamondsuit$

We now come to a key

Lemma 8 *There exists $D^* = [-l^*, l^*] \times [-l^*, l^*]$ such that if $D(l) \supseteq D^*$ and $\zeta \in \Sigma_{dss}(D(l))$, then supp($\zeta$) is not dense at either of the points $P_l := (l, 0)$ or $\underline{P_l} = (-l, 0)$.*

Proof. Since we are considering DSS-solutions, it suffices to prove the lemma only for P_l. To do this, we fix $R > a$ sufficiently large such that

$$\frac{1}{4\pi}||\zeta_0||_1^2 \log \frac{1}{R} < \Psi(\zeta_0^*). \tag{2.14}$$

Let $D := D(l) \supseteq B_R$ and suppose $\zeta \in \Sigma_{dss}(D)$. Define, $\psi_1 := K\zeta$. Next, we derive a lower bound for $\psi_1(x) - \psi_1(P_l)$. We do this in two steps. First, we find a lower bound for $\psi_1(0) - \psi_1(P_l)$ as follows

$$\psi_1(0) - \psi_1(P_l) = \frac{1}{2\pi} \int_D \log \frac{|P_l - y|}{|y|} \zeta(y) dy = \frac{1}{2\pi} \sum_{j=1}^{3} \int_{S_j} \log \frac{|P_l - y|}{|y|} \zeta(y) dy, \tag{2.15}$$

where $S_1 := \{x \in D | x_1 > 0\}$, $S_2 := \{x \in D | x_1 < 0,\ |x| < R\}$, and $S_3 := D \setminus S_1 \cup S_2$. Let us show that the integral over S_1 in (2.15) is positive, using methods in [40] which are similar to those in [27]. To do this we introduce the new sets

$$S_1^+ := \{x \in S_1 | x_1 > l/2\},\ S_1^- := \{x \in S_1 | x_1 < l/2\},\ \Omega = \text{supp}(\zeta) \cap S_1^+.$$

By Ω^* we mean the reflection of Ω about the line $x_1 = l/2$. Since ζ is DSS, it is immediate that

$$\int_\Omega \log \frac{|P_l - y|}{|y|} \zeta(y) dy \geq - \int_{\Omega^*} \log \frac{|P_l - y|}{|y|} \zeta(y) dy.$$

Therefore

$$\begin{aligned}\int_{S_1} \log \frac{|P_l - y|}{|y|} \zeta(y)dy &= \int_{S_1^+} \log \frac{|P_l - y|}{|y|} \zeta(y)dy + \int_{S_1^-} \log \frac{|P_l - y|}{|y|} \zeta(y)dy \\ &\geq \int_{S_1^- \setminus \Omega^*} \log \frac{|P_l - y|}{|y|} \zeta(y)dy > 0.\end{aligned}$$

Next we estimate the integral over S_2 in (2.15) to obtain

$$\int_{S_2} \log \frac{|P_l - y|}{|y|} \zeta(y)dy > \int_{S_2} \log \frac{l}{R} \zeta(y)dy = \frac{1}{2} \log \frac{l}{R}\, \beta(l), \tag{2.16}$$

where $\beta(l) := \int_{B_R} \zeta$, the circulation of the disc B_R. Finally, we estimate the integral over S_3 in (2.15)

$$\int_{S_3} \log \frac{|P_l - y|}{|y|} \zeta(y)dy > \int_{S_3} \log \frac{\sqrt{2}l}{l} \zeta(y)dy = 1/2 \log \sqrt{2}(\|\zeta_0\|_1 - \beta(l)). \tag{2.17}$$

Hence, from (2.15), (2.16) and (2.17) we infer

$$\psi_1(0) - \psi_1(P_l) > 1/4\pi(\log \frac{l}{\sqrt{2}R}\, \beta(l) + 1/2\|\zeta_0\|_1 \log \sqrt{2}). \tag{2.18}$$

Now, we consider the difference $\psi_1(x) - \psi_1(0)$, for $x \in \mathbb{R}^2$. Indeed, we have

$$\psi_1(x) - \psi_1(0) = \frac{1}{2\pi} \int_D \log \frac{|y|}{|x - y|} \zeta(y)dy = -\frac{1}{2\pi} \int_D \log \frac{|x| + |y|}{|y|} \zeta(y)dy.$$

Hence

$$\psi_1(x) - \psi_1(0) \geq -\frac{1}{2\pi} \int_{B_a} \log(1 + \frac{|x|}{|y|}) \zeta^*(y)dy. \tag{2.19}$$

Thus, by applying Hölder's inequality to (2.19), we deduce

$$\psi_1(x) - \psi_1(0) \geq -\frac{1}{2\pi} \left(\int_{B_a} \log^q \left(1 + \frac{|x|}{|y|}\right) dy \right)^{1/q} \|\zeta_0\|_p.$$

Therefore

$$\psi_1(x) - \psi_1(0) \geq -C|x|,\ \forall x \in \mathbb{R}^2, \tag{2.20}$$

where C is a positive constant depending on a and p. From (2.18) and (2.20), it follows

$$\begin{aligned}\psi_1(x) - \psi_1(P_l) - S/2\, x_2^2 \ &\geq \ 1/4\pi(\log \frac{l}{\sqrt{2}R}\, \beta(l) + 1/2\|\zeta_0\|_1 \log \sqrt{2}) \\ &\quad -C|x| - S/2\, |x|^2,\end{aligned}$$

for every $x \in \mathbb{R}^2$. Hence,

$$\psi(x) - \psi(P_l) \geq 1/4\pi \left(\log \frac{l}{\sqrt{2}R}\, \beta(l) + 1/2 \|\zeta_0\|_1 \log \sqrt{2} \right) - CR - S/2\, R^2, \qquad (2.21)$$

for every $x \in B_R$, where $\psi(x) = K\zeta(x) - S/2 x_2^2$. Now we state the following

Assertion. There exists $l' > R$ such that $\beta(l)$ is positively bounded away from zero, for all $l \geq l'$; that is,

$$\beta(l) \geq \alpha,\ \forall l \geq l',\ \text{and some } \alpha > 0.$$

Let us assume, for the moment, that the Assertion is true. Then from (2.21) we deduce

$$\psi(x) - \psi(P_l) \geq 1/4\pi (\log \frac{l}{\sqrt{2}R}\, \alpha + 1/2 \|\zeta_0\|_1 \log \sqrt{2}) - CR - S/2\, R^2,$$

for every $x \in B_R$ and all $l \geq l'$. Hence, it follows that there exists l^* such that

$$\psi(x) - \psi(P_l) > 0,\ \forall x \in B_R,\ \forall l \geq l^*. \qquad (2.22)$$

Finally, we set $D^* = [-l^*, l^*] \times [-l^*, l^*]$. To see that D^* indeed satisfies the property stated in the lemma, we fix $D \supseteq D^*$ and suppose $\zeta \in \Sigma_{dss}(D)$ and $\mathrm{supp}(\zeta)$ is dense at P_l. Since, $|B_R| > \pi a^2$, we can find $A \subset B_R \setminus \mathrm{supp}(\zeta)$ such that $|A| > 0$. Invoking (2.22) we infer

$$\psi(x) - \psi(P_l) > 0,\ \forall x \in A,$$

which is impossible by Lemma 7. Therefore, it remains to prove the Assertion.

Proof of the Assertion. Suppose the claim in the Assertion is not true. Then there exist sequences $\{l_n\}_{n=1}^{\infty}$ and $\{\zeta_n\}_{n=1}^{\infty}$ such that $l_n \to \infty$ as $n \to \infty$ and $\zeta_n \in \Sigma_{dss}(D(l_n))$, for all $n \in \mathbb{N}$, with the following property

$$\beta(l_n) := \int_{B_R} \zeta_n \to 0,\ \text{as } n \to \infty. \qquad (2.23)$$

We first show that for $E_n = \mathrm{supp}(\zeta_n) \cap B_R$ and $\gamma_n = |E_n|$, (2.23) implies

$$\gamma_n \to 0,\ \text{as } n \to \infty. \qquad (2.24)$$

Obviously, if (2.24) does not hold then there would exist a subsequence of $\{l_n\}_{n=1}^{\infty}$, again denoted $\{l_n\}_{n=1}^{\infty}$, such that

$$\gamma_n \geq \epsilon,\ \forall n \text{ and for some } \epsilon > 0.$$

Therefore

$$\int_{E_n} \zeta_n = -\int_{\mathbb{R}^2} -\chi_{E_n}\zeta_n \geq -\int_{\mathbb{R}^2} -\chi_{[\pi a^2-\gamma_n,\pi a^2]}\zeta_0^{\triangle},$$

where we have used (2.6). Therefore

$$\int_{B_R} \zeta_n \geq \int_{\pi a^2-\epsilon}^{\pi a^2} \zeta_0^{\triangle} > 0,$$

which is impossible by (2.23). Thus (2.24) holds. On the other hand, for $x \in \mathbb{R}^2$, we have the following estimate

$$\begin{aligned} K\zeta_n(x) &= \frac{1}{2\pi}\int_{B_R(x)} \log\frac{1}{|x-y|}\zeta_n(y)dy + \frac{1}{2\pi}\int_{\mathbb{R}^2\setminus B_R(x)} \log\frac{1}{|x-y|}\zeta_n(y)dy \\ &\leq \frac{1}{2\pi}\int_{B_R(x)} \log\frac{1}{|x-y|}\zeta_n(y)dy - \frac{1}{2\pi}\|\zeta_0\|_1 \log R. \end{aligned}$$

If we set $E_n(x) := \mathrm{supp}(\zeta_n) \cap B_R(x)$, an application of Hölder's inequality implies

$$\begin{aligned} \int_{B_R(x)} \log\frac{1}{|x-y|}\zeta_n(y)dy &\leq \left(\int_{B_R(x)} |\log|x-y||^q dy\right)^{1/q} \|\zeta_n\|_{p,E_n(x)} \\ &\leq C\|\zeta_n\|_{p,E_n(x)}, \end{aligned}$$

where C is a constant depending on R and p. Therefore

$$K\zeta_n(x) \leq -\frac{1}{2\pi}\|\zeta_0\|_1 \log R + C\|\zeta_n\|_{p,E_n(x)},\ \forall n,\ \forall x \in \mathbb{R}^2.$$

Next, observe that

$$\|\zeta_n\|_{p,E_n(x)} \leq \|\zeta_0^{\triangle}\|_{p,[0,\gamma_n(x)]},\ \forall n \in \mathbb{N},\ \forall x \in \mathbb{R}^2,$$

where $\gamma_n(x) := |E_n(x)|$. Moreover, since ζ_n's are DSS, we infer that

$$\gamma_n(x) \leq \gamma_n,\ \forall n \in \mathbb{N},\ \forall x \in \mathbb{R}^2.$$

Therefore we obtain

$$K\zeta_n(x) \leq -\frac{1}{2\pi}\|\zeta_0\|_1 \log R + C\|\zeta_0^{\triangle}\|_{p,[0,\gamma_n]},\ \forall n \in \mathbb{N},\ \forall x \in \mathbb{R}^2.$$

This, in turn, implies

$$\Psi(\zeta_n) \leq \frac{1}{2}\int_{\mathbb{R}^2} \zeta_n K \zeta_n \leq -\frac{1}{4\pi}\|\zeta_0\|_1^2 \log R + C\|\zeta_0\|_1 \|\zeta_0^{\triangle}\|_{p,[0,\gamma_n]}.$$

Now, since $\gamma_n \to 0$, as $n \to \infty$, we infer

$$\limsup_{n\to\infty} \Psi(\zeta_n) \leq -\frac{1}{4\pi}\|\zeta_0\|_1^2 \log R.$$

Hence, (2.14) implies

$$\limsup_{n\to\infty} \Psi(\zeta_n) < \Psi(\zeta_0^*).$$

Whence there exists $n_0 \in \mathbb{N}$ such that $\Psi(\zeta_{n_0}) < \Psi(\zeta_0^*)$. This is a contradiction to the maximality of ζ_{n_0}, since $\zeta_0^* \in \mathcal{F}(D(l_{n_0}))$. This completes the proof of the lemma. $\diamondsuit$

2.5 Proof of Theorem 2

We are now in a position to prove the main theorem.

Proof of Theorem 2. We will show that P has a DSS-solution. To do this, it suffices to find a square, say D_* such that for $D \supseteq D_*$ and $\zeta \in \Sigma_{dss}(D)$, $\text{supp}(\zeta) \subseteq D_*$. If D^* is the square constructed in Lemma 8, then it is clear that for $D \supseteq D^*$ and $\zeta \in \Sigma_{dss}(D)$, $\text{supp}(\zeta)$ lies in the strip $[-l^*, l^*] \times \mathbb{R}$. Hence, it remains to show that if $D := D(l)$ is sufficiently large and $\zeta \in \Sigma_{dss}(D)$, then $\text{supp}(\zeta)$ can not be dense at the points $Q_l := (0, l)$ and $\overline{Q_l} := (0, -l)$. Of course, since ζ is DSS, we only need to show the truth of this matter with respect to Q_l. Let us note that in the following D indicates a square containing $B_{2R} \cup D^*$, where $R > a$ is a constant satisfying (2.14). For such a $D := D(l)$ we consider ζ, a function in $\Sigma_{dss}(D)$. If $A := B_{2R} \setminus B_R$, then for $x \in A$ we have

$$\psi(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{1}{|x-y|}\zeta(y)dy - \frac{S}{2}x_2^2 \geq \frac{1}{2\pi}\log\frac{1}{2\sqrt{2}l}\,\|\zeta_0\|_1 - 2R^2 S. \tag{2.25}$$

On the other hand, in view of (2.9), we obtain

$$\psi(Q_l) \leq C\|\zeta_0\|_p - \frac{S}{2}l^2. \tag{2.26}$$

Therefore, from (2.25) and (2.26), it follows that if l is sufficiently large, say $l \geq l'$, then

$$\psi(x) > \psi(Q_l), \; \forall x \in A. \tag{2.27}$$

As in the proof of Lemma 8, equation (2.27) implies that if $l \geq l'$, then $\mathrm{supp}(\zeta)$ can not be dense at Q_l. Next we let $l_* := \max\{l^*, l'\}$ and set $Q_* := [-l_*, l_*] \times [-l_*, l_*]$, which is indeed the desired square. Therefore, any $\zeta \in \Sigma_{dss}(D_*)$ is a solution to P. This completes the existence part of the theorem. Now, let $\zeta \in \Sigma$, then there exists $D := D(l)$ which contains $\mathrm{supp}(\zeta)$ and obviously $\zeta \in \Sigma(D)$. Moreover, by Lemma 4 we have

$$\zeta = \phi \circ (K\zeta - \frac{S}{2}x_2^2), \text{ a.e. in } D, \tag{2.28}$$

where ϕ is an increasing function, unknown a priori. We need to modify ϕ in order to obtain a similar equation to (2.28) which holds almost everywhere in $\mathbb{R}^2$. To do this, we first note that since $\psi := K\zeta - \frac{S}{2}x_2^2$ belongs to $C^1(\mathbb{R}^2)$, it attains its minimum, say γ, relative to the closure of D. On the other hand,

$$\psi(x) \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-y|} \zeta(y) dy \leq \frac{1}{2\pi} \|\zeta_0\|_1 \log \frac{2}{|x|},$$

for every $x \in \mathbb{R}^2$ satisfying $|x| > 1 + 2\sqrt{2}l$. This, in turn, implies that there exists $D_1 \supseteq D$ for which the following holds

$$\psi(x) < \gamma, \; \forall x \in \mathbb{R}^2 \setminus D_1.$$

Since $\zeta \in \Sigma(D_1)$, another application of Lemma 4 ensures the existence of an increasing function, say ϕ_1, such that

$$\zeta = \phi_1 \circ \psi, \text{ a.e. in } D_1.$$

We now define

$$\phi^\dagger(t) = \begin{cases} \phi_1(t) & t \geq \gamma \\ 0 & t < \gamma. \end{cases}$$

Therefore, $\zeta = \phi^\dagger \circ \psi$, almost everywhere in $\mathbb{R}^2$. Hence, applying Lemma 2, we obtain

$$-\Delta\psi_1 = \phi^\dagger \circ (\psi_1 - \frac{S}{2}x_2^2), \text{ a.e. in } \mathbb{R}^2.$$

This completes the proof of the theorem.

Chapter 3

A constrained variational problem for steady vortices in a shear flow

3.1 Introduction

In this chapter we study existence of solutions for a family of maximisation problems $P(I)$, $I > 0$, where the set of admissible functions comprises all rearrangements of a given non-negative function which satisfy an equality constraint, which we call the impulse condition. Physically, the maximisers will enable us to model a special class of steady 2-dimensional ideal fluid flows. These flows enjoy the property that each is separated into two regions of positive constant and of positive non-constant vorticity. The former region is bounded while the later is unbounded. It is always possible to arrange for the free boundary to be symmetric about the two co-ordinate axes. Moreover, these flows behave like uniform shear flows with non-negative strength, at infinity. The variational principle is adapted from one for vortex rings in 3-dimensions, proposed by Benjamin [6]. However, application of this method presents two mathematical difficulties. Firstly, a lack of compactness arises from the non-convexity of the set of rearrangements, regarded as a subset of an L^p space. Secondly, a lack of compactness is caused by the unboundedness of the domain of interest $\mathbb{R}^2$. Therefore, we first consider $P(I)$ in a bounded square D, denoting the modified problem $P(I, D)$. Then by introducing an auxiliary problem $P(I, \leq, D)$ with an inequality constraint together with results from Burton [10, 11] we show that $P(I, D)$ is solvable for sufficiently large D. We then show, as anticipated in [6], that the maximisers are the same for all sufficiently large D, so consideration of a sufficiently large square yields a solution valid throughout $\mathbb{R}^2$.

This chapter builds upon recent work of Nycander [40] where a slightly different variational principle for a vortex anomaly in shear flows was employed; this was later extended by Emamizadeh [21].

3.2 Notation, definitions and statement of results

D will always denote a square of the form $[-l,l]\times[-l,l]$. In particular, $D_0=[-l_0,l_0]\times[-l_0,l_0]$, where $l_0=\sqrt{\pi}a$ for some fixed $a>0$. In case we need to emphasize the length l we write $D(l)$. The symbols x, y, etc. denote points in $\mathbb{R}^2$, hence $x=(x_1,x_2)$, $y=(y_1,y_2)$, etc. Throughout p will be a real number greater than 2. We use $|A|$ to denote the 2-dimensional Lebesgue measure (or area) of a set $A\subset\mathbb{R}^2$. Moreover, any square D will satisfy the condition $|D|\geq\pi a^2$, hence $l\geq\sqrt{\pi}a$. Let $\zeta_0\in L^p(\mathbb{R}^2)$ be a non-negative function which vanishes outside a set of measure πa^2. The set of all rearrangements of ζ_0 on $\mathbb{R}^2$ which vanish outside bounded sets is denoted by $\mathcal{F}$. The subset of $\mathcal{F}$ comprising functions vanishing outside the square D is denoted $\mathcal{F}(D)$. For a non-negative $\zeta\in L^p(\mathbb{R}^2)$ having compact support, we define the *energy functional*

$$\Psi(\zeta)\;=\;\frac{1}{2}\int_{\mathbb{R}^2}\zeta K\zeta, \tag{3.1}$$

where

$$K\zeta(x)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\log\frac{1}{|x-y|}\zeta(y)dy. \tag{3.2}$$

It will be recalled that the operator K defined by (3.2) maps $L^p(U)$ into $L^q(U)$, and is moreover compact, provided $q\geq 1$ and U is a bounded subset of $\mathbb{R}^2$, see [Lemma 2, Chapter 2]. Hence the energy Ψ defined in (3.1) is meaningful. Again for non-negative $\zeta\in L^p(\mathbb{R}^2)$ with compact support, we define the *impulse functional*

$$\Im(\zeta)=\int_{\mathbb{R}^2}x_2^2\,\zeta. \tag{3.3}$$

For any positive I, $P(I)$ denotes the following maximisation problem:

$$P(I):\quad\sup_{\zeta\in\mathcal{F},\Im(\zeta)=I}\Psi(\zeta),$$

and the set of solutions of $P(I)$ is denoted by $\Sigma(I)$. Similarly, for a square D we define $P(I,D)$ as follows

$$P(I,D):\quad\sup_{\zeta\in\mathcal{F}(D),\Im(\zeta)=I}\Psi(\zeta),$$

and the set of solutions of $P(I,D)$ is denoted by $\Sigma(I,D)$. Note that in $P(I)$ the set $\mathcal{F}\cap\Im^{-1}(I)$ is not empty for any positive I. However, in $P(I,D)$ we must assume D is sufficiently large to ensure $\mathcal{F}(D)\cap\Im^{-1}(I)$ is not empty. We will also be dealing with maximisation problems with inequality constraints, namely

$$P(I,\leq,D): \sup_{\zeta\in\mathcal{F}(D),\Im(\zeta)\leq I} \Psi(\zeta),$$

where again it is assumed that D is sufficiently large to ensure $P(I,\leq,D)$ is meaningful. The set of solutions of $P(I,\leq,D)$ is denoted by $\Sigma(I,\leq,D)$. We are now ready to state the main theorem.

Theorem *For any positive real number I, $P(I)$ has a solution, that is, $\Sigma(I)\neq\emptyset$. Moreover, if $\zeta\in\Sigma(I)$ then $\psi:=K\zeta$ will satisfy the following semilinear elliptic partial differential equation*

$$-\Delta\psi=\phi\circ(\psi-\lambda x_2^2), \text{ a.e. in } \mathbb{R}^2, \tag{3.4}$$

where ϕ is an increasing function and λ is a non-negative real number, both unknown a priori. Furthermore, the maximiser ζ can be chosen to be doubly Steiner-symmetric (see the definition below).

This theorem should be compared with the main result of Nycander [40], who proved the existence of a maximiser for $\Psi-\lambda\Im$ relative to $\mathcal{F}$, where $\lambda>0$ is prescribed and the value of $\Im$ at the maximiser is unknown. It may be shown (see Emamizadeh [21]) that this again gives rise to a solution for (3.4). Our result maximises Ψ relative to $\mathcal{F}$ with a prescribed value of $\Im$, and λ is an unknown "Lagrange multiplier".

Physically, ζ represents a vorticity anomaly imposed on an ambient uniform shear flow of strength 2λ; the functional Ψ represents kinetic energy, and $\psi-\lambda x_2^2$ is the stream function for the flow. There does not appear a satisfactory physical interpretation of $\Im$. Nycander's work was motivated by equatorial flows on giant planets. The corresponding problems with a uniform ambient flow, instead of shear flow, were considered by Burton [12]. In his work, existence was only proved for small λ and large $\Im$, whereas for the shear flow these restrictions are unnecessary. The shear problem is solved by a strong compactness argument (following Nycander [40]) in contrast with Burton's use of weak compactness.

In this chapter, similarly to Chapter 2, we make use of Steiner, Schwarz and decreasing rearrangements of functions. The reader is referred to Chapter 1 for relevant definitions and notation. In particular we recall that if $h\in L^p(\mathbb{R}^2)$ and $g\in L^q(\mathbb{R}^2)$, where q is the conjugate exponent of p, and h and g vanish outside a set of measure ω

then

$$\int_{\mathbb{R}^2} hg \leq \int_0^\omega h^\triangle g^\triangle. \tag{3.5}$$

Also, if f is a non-negative, measurable function on $\mathbb{R}^2$, then

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x)\log\frac{1}{|x-y|}f(y)dxdy \leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^*(x)\log\frac{1}{|x-y|}f^*(y)dxdy \tag{3.6}$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x)\log\frac{1}{|x-y|}f(y)dxdy \leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^\sharp(x)\log\frac{1}{|x-y|}f^\sharp(y)dxdy \tag{3.7}$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x)\log\frac{1}{|x-y|}f(y)dxdy \leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\sharp(x)\log\frac{1}{|x-y|}f_\sharp(y)dxdy. \tag{3.8}$$

We denote by χ_A the characteristic function of a measurable set A; that is

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A. \end{cases}$$

For a measurable function ζ, $\mathrm{supp}(\zeta)$ denotes the *strong support of* ζ, that is,

$$\mathrm{supp}(\zeta) = \{x \in \mathbb{R}^2 | \zeta(x) > 0\}.$$

3.3 Preliminary results

In this section we will derive some properties of K, Ψ and $\Im$ which will be used throughout the chapter.

3.3.1 Properties of K

We begin by recalling some results from chapter 2, concerning the operator K, which are included here for the sake of completeness.

Lemma 1 *Let $\zeta \in L^p(\mathbb{R}^2)$ and vanish outside a bounded set. Then $K\zeta \in C^1(\mathbb{R}^2)$. Moreover, the following holds*

$$\nabla K\zeta(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}\zeta(y)dy, \ \forall x \in \mathbb{R}^2.$$

Moreover, we have the following estimate

$$|\nabla K\zeta(x)| \leq M_2\|\zeta\|_p, \ \forall x \in \mathbb{R}^2, \tag{3.9}$$

where M_2 depends on $|\mathrm{supp}(|\zeta|)|$.

Lemma 2 *(i) If $\zeta \in L^p(\mathbb{R}^2)$ vanishes outside a bounded set in $\mathbb{R}^2$ then $K\zeta$ is defined everywhere in $\mathbb{R}^2$ and $K\zeta \in W^{2,p}_{loc}(\mathbb{R}^2)$. Moreover, we have*

$$-\Delta K\zeta = \zeta, \ \text{a.e. in}\, \mathbb{R}^2.$$

(ii) If U is a bounded open set in $\mathbb{R}^2$ then $K : L^p(U) \to H^1(U)$ is continuous and hence in view of the compact embedding $H^1(U) \hookrightarrow\hookrightarrow L^q(U)$, $q \geq 1$, the mapping $K : L^p(U) \to L^q(U)$ is a linear compact transformation .

(iii) K is not positive in the sense of operators; that is the quadratic Ψ is indefinite.

(iv) K is symmetric i.e. for every ζ_1, $\zeta_2 \in L^p(\mathbb{R}^2)$ vanishing outside bounded sets the following holds

$$\int_{\mathbb{R}^2} \zeta_1 K\zeta_2 = \int_{\mathbb{R}^2} \zeta_2 K\zeta_1.$$

3.3.2 Properties of Ψ and $\Im$

Lemma 3 *(i) If $\zeta \in L^p(\mathbb{R}^2)$ vanishes outside a bounded set then $\Psi(\zeta)$ is finite.*

(ii) For every $\zeta \in \mathcal{F}$, $\Psi(\zeta) \leq \Psi(\zeta^*) = \Psi(\zeta_0^*)$.

(iii) For every $\zeta \in \mathcal{F}$, $\Psi(\zeta_t) = \Psi(\zeta^t) = \Psi(\zeta)$, where $\zeta_t(x_1, x_2) = \zeta(x_1 - t, x_2)$ and $\zeta^t(x_1, x_2) = \zeta(x_1, x_2 - t)$, $\forall t \in \mathbb{R}$. In other words , the energy is translation-invariant.

(iv) If $\zeta \in L^p(\mathbb{R}^2)$ vanishes outside a bounded set then $\Psi(\zeta) \leq \Psi(\zeta^\sharp)$ and $\Psi(\zeta) \leq \Psi(\zeta_\sharp)$.

(v) For any D, $\Psi \in C^1(L^p(D))$.

(vi) For a square D, Ψ is Lipschitz on any bounded subset of $L^p(D)$, in particular, Ψ is Lipschitz on $\overline{\mathcal{F}(D)}^w$, the weak closure of $\mathcal{F}(D)$ in $L^p(D)$.

Proof. (i) follows from the fact that $K\zeta \in W^{1,\infty}_{loc}$. (ii) follows from (3.6). (iii) is trivial. (iv) follows from (3.7) and (3.8). To prove (v) we take ζ_1, $\zeta_2 \in L^p(\mathbb{R}^2)$ such that both vanish outside D. Since K is symmetric we deduce

$$\Psi(\zeta_1 + \zeta_2) = \Psi(\zeta_1) + \Psi(\zeta_2) + \int_{\mathbb{R}^2} \zeta_2 K\zeta_1. \tag{3.10}$$

On the other hand by applying Hölder's inequality and invoking the boundedness of K we obtain

$$\Psi(\zeta_2) = o(||\zeta_2||), \text{ as } ||\zeta_2||_p \to 0.$$

This shows that Ψ is Frechet differentiable and if we denote its derivative by Ψ' then from (3.10) we obtain

$$\Psi'(\zeta)(\overline{\zeta}) = \int_{\mathbb{R}^2} \overline{\zeta} K \zeta.$$

Boundedness of K and Hölder's inequality now show

$$\|\Psi'(\zeta)\| \leq \|K\| \, \|\zeta\|_p, \; \forall \zeta \in L^p(D).$$

Therefore $\|\Psi'\| \leq \|K\|$, hence $\Psi \in C^1(L^p(D))$ as required. Finally, (vi) follows from (v) and the Mean Value Inequality. $\diamondsuit$

Lemma 4 *(i) If $\zeta \in \mathcal{F}$ and $t \in \mathbb{R}$ then $\Im(\zeta_t) = \Im(\zeta)$; that is, $\Im$ is invariant under translations parallel to x_1-axis. Moreover, if ζ is even in x_2 then $\Im(\zeta^t) \geq \Im(\zeta)$.*
(ii) If $\zeta \in \mathcal{F}$ then $\Im(\zeta^\sharp) = \Im(\zeta)$ and $\Im(\zeta_\sharp) \leq \Im(\zeta)$.

Proof. (i) is trivial. To prove (ii) we first note that from Fubini's theorem we immediately obtain $\Im(\zeta^\sharp) = \Im(\zeta)$. For the second part we apply a classical inequality of Hardy, see for example [34],

$$\Im(\zeta) = -\int_{\mathbb{R}^2} -x_2^2 \zeta(x) dx \geq -\int_{\mathbb{R}^2} (-x_2^2)_\sharp \zeta_\sharp(x) dx = \Im(\zeta_\sharp).$$

Hence we are done. $\diamondsuit$

3.3.3 Some results from Burton's theory

Lemma 5 *Let D be a square. Then*
(i) $\overline{\mathcal{F}(D)}^w$ is weakly sequentially compact, and convex.
(ii) $\Im(\overline{\mathcal{F}(D)}^w) = \Im(\mathcal{F}(D)) = [a,b]$ for some a and b non-negative.
(iii) $ext(\overline{\mathcal{F}(D)}^w \cap W) = \mathcal{F}(D) \cap W$, for any affine subspace of $L^p(D)$ of finite co-dimension, where $ext(\overline{\mathcal{F}(D)}^w \cap W)$ denotes the set of extreme points of $\overline{\mathcal{F}(D)}^w \cap W$.
(iv) $\overline{\mathcal{F}(D) \cap \Im^{-1}(I)}^w = \overline{\mathcal{F}(D)}^w \cap \Im^{-1}(I)$.

Proof. See Burton [11] for (i), Burton and Ryan [15] for (ii) and (iii). To prove (iv) we first note that since $\overline{\mathcal{F}(D)}^w \cap \Im^{-1}(I)$ is weakly closed, $\overline{\mathcal{F}(D) \cap \Im^{-1}(I)}^w \subset \overline{\mathcal{F}(D)}^w \cap \Im^{-1}(I)$. To show the reverse inclusion we fix $\eta \in \overline{\mathcal{F}(D)}^w \cap \Im^{-1}(I)$. Next we let U be a basic weakly open set containing η, hence there exist $\epsilon > 0$ and functions $f_1, f_2, \ldots, f_k$ in $L^q(D)$ such that

$$U = \{\zeta \in L^p(D) \,|\, |f_j(\zeta) - f_j(\eta)| \leq \epsilon, \; j = 1, 2, \ldots, k\},$$

where q is the conjugate exponent of p. Next we consider a particular affine subspace of $L^p(D)$ with co-dimension k, namely

$$V = \{\zeta \in L^p(D) |\ f_j(\zeta) = f_j(\eta),\ j = 1, 2, \ldots, k\}$$

Then $\eta \in V$. Now by the Krein-Milman theorem

$$\overline{\mathcal{F}(D)^w} \cap \Im^{-1}(I) \cap V = \overline{co}(ext(\overline{\mathcal{F}(D)^w} \cap \Im^{-1}(I) \cap V)).$$

Therefore, $ext(\overline{\mathcal{F}(D)^w} \cap \Im^{-1}(I) \cap V) \neq \emptyset$. Applying (iii) we infer $\mathcal{F}(D) \cap \Im^{-1}(I) \cap V \neq \emptyset$. In particular, $\mathcal{F}(D) \cap \Im^{-1}(I) \cap U \neq \emptyset$. Hence $\eta \in \overline{\mathcal{F}(D) \cap \Im^{-1}(I)}^w$, and we are done. $\diamondsuit$

The following lemma is a simple version of [11, Corollary 3.4].

Lemma 6 *(local maximisers)Let O be an open, bounded set in $\mathbb{R}^2$. Let*

$$\mathcal{L}u := \sum_{1 \leq |\alpha| \leq m} \mathcal{A}^\alpha(x)\, \mathcal{D}^\alpha u$$

define a linear partial differential operator in O, where $\mathcal{A}^\alpha$ are measurable and there is no 0-th order term. Let $2 < p < \infty$ and p^* be the conjugate exponent of p. Let $\check{K} : L^p(O) \to L^{p^*}(O)$ be a compact, symmetric, linear operator and suppose $\check{K}\zeta \in W^{m,1}_{loc}(O)$, $\mathcal{L}\check{K}\zeta = \zeta$ almost everywhere in O for all $\zeta \in L^p(O)$. Define

$$\check{\Psi}(\zeta) := \int_O \zeta\, \check{K}\zeta,$$

for all $\zeta \in L^p(O)$. Let $\zeta_0 \in L^p(O)$ be non-negative and let $\mathcal{F}(O)$ be the set of rearrangements of ζ_0 on O. Suppose $\hat{\zeta} \in \mathcal{F}(O)$ and U is a strong neighbourhood of $\hat{\zeta}$ relative to $\mathcal{F}(O)$ such that $\check{\Psi}(\zeta) \leq \check{\Psi}(\hat{\zeta})$ for all $\zeta \in U$. If $\check{\psi} := \check{K}\hat{\zeta}$, then $\mathcal{L}\check{\psi} = \check{\phi} \circ \check{\psi}$ almost everywhere in O, for some increasing function $\check{\phi}$.

3.4 Some auxiliary lemmas

Lemma 7 *If $\zeta \in L^p(\mathbb{R}^2)$, vanishing outside D, is DSS then $\psi := K\zeta$ is also DSS.*

Proof. Let us fix ζ as in the lemma. Recall that for such a function, $\psi \in C^1(\mathbb{R}^2)$ and

$$\nabla\psi(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \zeta(y) dy.$$

We only show that ψ is Steiner-symmetric about the line $x_1 = 0$, the other case $x_2 = 0$

being identical. For $x \in \mathbb{R}^2$ we let $\underline{x}$ denote the reflection of x with respect to the x_2-axis: $\underline{x} = (-x_1, x_2)$. Now for $x \in \mathbb{R}^2$ we have

$$\begin{aligned}
\psi(\underline{x}) &:= \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{1}{|\underline{x}-y|}\zeta(y)dy = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{1}{|x-\underline{y}|}\zeta(y)dy \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{1}{|x-y|}\zeta(\underline{y})dy = \psi(x).
\end{aligned}$$

Hence ψ is even in the x_1-variable. Next we show that for fixed x_2, $\psi(x_1, x_2)$ is non-increasing for $x_1 \geq 0$. The argument to follow is independent of the choice of x_2, hence for simplicity we set $x_2 = 0$. Then ψ is even in the x_1-variable. Since we have

$$\frac{\partial}{\partial x_1}\psi(x_1, 0) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x_1 - y_1}{|(x_1,0)-y|^2}\zeta(y)dy,$$

it suffices to prove the function

$$I(\alpha) = \int_{\mathbb{R}^2} \frac{\alpha - y_1}{|(\alpha,0)-y|^2}\zeta(y)dy$$

is non-negative for all $\alpha \geq 0$; this obvious for $\alpha \geq l$. To do this, we fix α and denote the reflection of $x \in \mathbb{R}^2$ about the line $x_1 = \alpha$ by x_α, that is, $x_\alpha = (2\alpha - x_1, x_2)$. Then we have

$$\begin{aligned}
I(\alpha) &= \int_{-l}^{l}\int_{-l}^{2\alpha-l} \frac{\alpha - y_1}{|(\alpha,0)-y|^2}\zeta(y)dy_1dy_2 + \int_{-l}^{l}\int_{2\alpha-l}^{\alpha} \frac{\alpha - y_1}{|(\alpha,0)-y|^2}\zeta(y)dy_1dy_2 \\
&+ \int_{-l}^{l}\int_{\alpha}^{l} \frac{\alpha - y_1}{|(\alpha,0)-y|^2}\zeta(y)dy_1dy_2 = I_1 + I_2 + I_3
\end{aligned}$$

where $I_j, j = 1,2,3$ have the obvious definitions. We observe that if $-l < y_1 < 2\alpha - l$ then $\alpha - y_1 > 0$, hence $I_1 \geq 0$. Moreover, by a change of variable we get

$$I_2 = -\int_{-l}^{l}\int_{\alpha}^{l} \frac{\alpha - y_1}{|(\alpha,0)-y_\alpha|^2}\zeta(y_\alpha)dy_1dy_2.$$

Now for $\alpha < y_1 < l$ we have $-y_1 < 2\alpha - y_1 < y_1$ so $\zeta(y_\alpha) \geq \zeta(y)$, since ζ is even and decreasing. Therefore

$$I_2 + I_3 = \int_{-l}^{l}\int_{\alpha}^{l} \frac{\alpha - y_1}{|(\alpha,0)-y_\alpha|^2}(\zeta(y) - \zeta(y_\alpha))dy_1dy_2 \geq 0.$$

We now have $I_\alpha \geq 0$ for $0 \leq \alpha \leq l$ as required.◇

The next lemma is a result of Nycander [40].

Lemma 8 *Let D be a square and $k > 0$. Then the set $W_k := W = \{\zeta |\ 0 \leq \zeta \leq k$ a.e. in D, ζ is $DSS\}$ is totally bounded in $L^1(D)$.*

Lemma 9 *Let $\{\zeta_n\}_{n=1}^{\infty}$ be a DSS-sequence in $\mathcal{F}(D) \cap L^p(D)$, for some D. Then $\{\zeta_n\}_{n=1}^{\infty}$ is totally bounded in $L^p(D)$.*

Proof. Fix $\epsilon > 0$. Let ξ_h denote the *truncation-function* at height $h > 0$; in other words,

$$\xi_h(t) := \begin{cases} h & \text{if } t \geq h \\ t & \text{otherwise,} \end{cases}$$

and $\zeta_{n,h} := \xi_h \circ \zeta_n$. Clearly, there is a natural number n_0 such that $||\zeta_0^* - \zeta_{0,n_0}^*||_p \leq 3^{-1}\epsilon$. On the other hand the set $\{\zeta \in L^p(D) | \zeta$ is DSS and $0 \leq \zeta \leq n_0\}$ is totally bounded, by Lemma 8. This, in turn, ensures the set $\{\zeta_{n,n_0} | n \in \mathbb{N}\}$ must also be totally bounded. Thus we can find natural numbers $n_1, \ldots, n_k$ such that for every natural number n there exists n_i, for some $i \in \{1, 2, \ldots, k\}$, for which

$$||\zeta_{n,n_0} - \zeta_{n_i,n_0}||_1 \leq \frac{\epsilon^p}{3^p(2n_0)^{\frac{p}{q}}}, \tag{3.11}$$

where q is the conjugate exponent of p. On the other hand

$$||\zeta_{n,n_0} - \zeta_{n_i,n_0}||_p^p = \int_{\mathbb{R}^2} |\zeta_{n,n_0} - \zeta_{n_i,n_0}|^{p-1} |\zeta_{n,n_0} - \zeta_{n_i,n_0}| \leq (2n_0)^{p-1} ||\zeta_{n,n_0} - \zeta_{n_i,n_0}||_1,$$

hence from (3.11) we obtain $||\zeta_{n,n_0} - \zeta_{n_i,n_0}||_p \leq \frac{\epsilon}{3}$. Now let us fix n, then

$$||\zeta_n - \zeta_{n,n_0}||_p^p = \int_{\mathbb{R}^2} (\zeta_n - \zeta_{n,n_0})^p = \int_{\mathbb{R}^2} (\zeta_n - n_0)_+^p,$$

where "+" indicates the positive part of a function. Since ζ_n, ζ_n^* are in $\mathcal{F}(D)$ we also have

$$\int_{\mathbb{R}^2} (\zeta_n^* - n_0)_+^p = \int_{\mathbb{R}^2} (\zeta_n - n_0)_+^p.$$

But $\zeta_n^* = \zeta_0^*$ a.e in $\mathbb{R}^2$, hence

$$\int_{\mathbb{R}^2} (\zeta_n^* - n_0)_+^p = \int_{\mathbb{R}^2} (\zeta_0^* - n_0)_+^p = ||\zeta_0^* - \zeta_{0,n_0}^*||_p^p \leq (\frac{\epsilon}{3})^p.$$

Therefore we have

$$||\zeta_n - \zeta_{n,n_0}||_p \leq \frac{\epsilon}{3}.$$

Finally, from the triangle inequality and above calculations we deduce

$$\|\zeta_n - \zeta_{n_i}\|_p \leq \|\zeta_n - \zeta_{n,n_0}\|_p + \|\zeta_{n,n_0} - \zeta_{n_i,n_0}\|_p + \|\zeta_{n_i,n_0} - \zeta_{n_i}\|_p \leq \epsilon.$$

This proves the set V is totally bounded. $\diamondsuit$

3.5 Proof of the theorem

The proof of the theorem emerges from a series of lemmas, but first we make the following three remarks

Remark 1. The theorem displays an example of an optimisation problem with a *lack of compactness.* Therefore, the usual direct method of calculus of variations cannot be applied without modification. Specifically, let $\{\zeta_n\}_{n=1}^{\infty}$ be a maximising sequence. Then for $t > 0$ the sequence $\{(\zeta_n)_t\}_{n=1}^{\infty}$ will also be a maximising sequence, where the subscript t indicates translation in the x_1 direction. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence such that $t_n \to \infty$, as $n \to \infty$. Then the sequence $\{(\zeta_n)_{t_n}\}_{n=1}^{\infty}$, which is a maximising sequence, converges to zero weakly, that is,

$$(\zeta_n)_{t_n} \overset{L^p}{\rightharpoonup} 0 \text{ as } n \to \infty$$

However, $0 \notin \mathcal{F}$, since ζ_0 is assumed to be non-zero. Therefore there is no hope of proving that every maximising sequence has a subsequence that converges to a maximiser.

Remark 2. If I in the theorem lies in the interval $[\Im(\zeta_0^*), \infty)$, then the existence part will become trivial, since in this case a solution would be obtained by translating ζ_0^* along the x_2 direction. However, even in this case, it is not trivial that there should be a differential equation of the form stated in the theorem.

According to Remark 2 our main concern in proving the theorem is the case $I < \Im(\zeta_0^*)$. For simplicity of notation we denote the interval $\Im(\overline{\mathcal{F}(D)^w}) \setminus [\Im(\zeta_0^*), \infty)$ by I_D, provided $D \supseteq D_0$. The interior of I_D is denoted $\mathrm{int}(I_D)$.

Remark 3. Note that it is always possible to ensure $\mathrm{int}(I_D) \neq \emptyset$ by making D large enough. To see this let us assume $0 < \alpha < 1$ and consider a square $D(l)$ with $l \geq a\alpha^{-1}$. Then by squashing and stretching ζ_0^* in an appropriate way one can guarantee that $\mathrm{int}(I_D) \neq \emptyset$. Specifically, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation

$$T(x_1, x_2) = (\alpha x_1, \alpha^{-1} x_2).$$

Clearly T is measure-preserving, hence $\zeta_0^* \circ T \in \mathcal{F}(D)$. Moreover, we obtain

$$\Im(\zeta_0^* \circ T) = \alpha^2 \Im(\zeta_0^*) < \Im(\zeta_0^*),$$

since $\alpha < 1$. Therefore $\Im(\zeta_0^* \circ T) \in \mathrm{int}(I_D)$.

Lemma 10 *Let $D \supseteq D_0$ and $I > 0$ such that $\mathcal{F}(D) \cap \Im^{-1}(0, I] \neq \emptyset$, then $\Sigma(I, \leq, D) \neq \emptyset$.*

Proof. We apply the direct method. To this end, let $\{\zeta_n\}_{n=1}^{\infty}$ be a maximising sequence. Invoking equations (3.7), (3.8) and Lemma 3(iv) it is clear that we may assume the ζ_n's are DSS. By Lemma 9 the sequence $\{\zeta_n\}_{n=1}^{\infty}$ is totally bounded in $L^p(D)$. Hence it contains a Cauchy subsequence, say $\{\zeta_{n_j}\}_{j=1}^{\infty}$. Since $L^p(D)$ is complete, there exists a function $\zeta \in L^p(D)$ such that $\zeta_{n_j} \to \zeta$, in $L^p(D)$, as $j \to \infty$. Moreover, $\zeta \in \mathcal{F}(D)$, since $\mathcal{F}(D)$ is strongly closed in $L^p(D)$. On the other hand, since $x_2^2 \in L^q(D)$, where q is the conjugate exponent of p, $\Im(\zeta_{n_j}) \to \Im(\zeta)$ as $j \to \infty$. Finally, continuity of Ψ ensures $\Psi(\zeta_{n_j}) \to \Psi(\zeta)$ as $j \to \infty$. Whence $\zeta \in \Sigma(I, \leq, D)$ and we are done. $\diamondsuit$

The next lemma is a special case of Fraenkel [24, Theorem 3.3 and Lemma 3.11], which are based on [27]. Fraenkel's result has the conclusion that a solution of a certain boundary value problem is symmetric; by assuming the solution to be DSS we are able to formulate a version where the solution is Schwarz symmetric about the origin. First we state a definition

Definition. Let $u \in C^1(\mathbb{R}^2)$. We shall say that u has *admissible asymptotic behaviour* if and only if the following condition holds outside some ball in $\mathbb{R}^2$: For some positive constants κ, ε and $\delta \in (0, 1]$,

$$u(x) = \kappa \log \frac{\varepsilon}{|x|} + h(x),$$

where $|\nabla h(x)| = O(|x|^{-2-\delta})$ and $h(x) \to 0$, as $|x| \to \infty$.

Remark. If $f \in L^p(\mathbb{R}^2)$ vanishes outside a bounded set, then Kf will have admissible asymptotic behaviour, see [24, Appendix A].

Lemma 11 *Suppose $u : \mathbb{R}^2 \to \mathbb{R}$ is DSS, belongs to $C^1(\mathbb{R}^2)$, and has admissible* *asymptotic behaviour. Moreover, suppose*

$$\int_{\mathbb{R}^2}(-\nabla\phi\cdot\nabla u+\phi\, f(u))=0,\ \forall\phi\in C_0^\infty(\mathbb{R}^2),$$

where f has the decomposition $f=f_1+f_2$ such that $f_1:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz continuous while $f_2:\mathbb{R}\to\mathbb{R}$ is increasing. Then u is a spherically symmetric decreasing function i.e. $u=u^*$.

Lemma 12 *Let $0<I<\Im(\zeta_0^*)$, and let $D_1=[-l,l]\times[-l,l]$ where l is chosen large enough that $D_1\supseteq D_0$ and $I\in int(I_{D_1})$. Then there exists $D_*\supseteq D_1$ such that any DSS-solution of $P(I,\leq,D)$, $D\supseteq D_*$, is a solution of $P(I,D)$.*

Proof. Define

$$V=\{\zeta|\ \zeta\in\Sigma(I,\leq,D)\text{ for some }D\supset D_1\text{ and }\zeta\text{ is }DSS\},$$

and consider $\zeta\in V$. Then there exists $D\supseteq D_1$ such that $\zeta_D:=\zeta\in\Sigma(I,\leq,D)$. We set $\psi_D:=K\zeta_D$. By maximality of ζ_D we infer $\Psi(\zeta_1)\leq\Psi(\zeta_D)$, where $\zeta_1\in\Sigma(I,\leq,D_1)$ is chosen arbitrarily. By Lemma 7, ψ_D is DSS, hence

$$\Psi(\zeta_D):=\frac{1}{2}\int_{\mathbb{R}^2}\zeta_D\psi_D\leq\frac{1}{2}||\zeta_0||_1\ \psi_D(0).$$

Therefore,

$$\Psi(\zeta_1)\leq\frac{1}{2}||\zeta_0||_1\psi_D(0),$$

so

$$\psi_D(0)\geq 2||\zeta_0||_1^{-1}\Psi(\zeta_1):=M_1.$$

Next, by an application of the Mean Value Inequality and in view of the estimate for $\nabla\psi_D(x)$ in Lemma 1, we obtain

$$\psi_D(x)\geq\psi_D(0)-M_2\sqrt{2}l,\ \forall x\in D_1,$$

where the positive constant M_2 is independent of D. From the above calculations we deduce

$$\psi_D(x)\geq M_1-M_2\sqrt{2}l:=\gamma,\ \forall x\in D_1.$$

Next we find an upper bound for ψ_D. To this end, let $x\in\mathbb{R}^2$ be such that $|x|>1+2\sqrt{2}l$; then for $y\in D_1$ we have $|x-y|\geq 2^{-1}|x|$. Now we decompose $\psi_D(x)$ as follows:

$$\psi_D(x)\ =\ \frac{1}{2\pi}\int_{D_1}\log\frac{1}{|x-y|}\zeta_D(y)dy+\frac{1}{2\pi}\int_{B_1(x)}\log\frac{1}{|x-y|}\zeta_D(y)dy$$

$$+\frac{1}{2\pi}\int_{\mathbb{R}^2\setminus D_1\cup B_1(x)}\log\frac{1}{|x-y|}\zeta_D(y)dy = I_1+I_2+I_3,$$

where I_1, I_2, I_3 have the obvious meaning. Clearly $I_3 \leq 0$, and

$$I_1 \leq (2\pi)^{-1}\|\zeta_0\|_1 \log 2|x|^{-1},$$

since $|x-y| \geq 2^{-1}|x|$, for all $y \in D_1$. Finally, by applying Hölder's inequality we obtain $I_2 \leq M_3$, where M_3 is a constant independent of x. Therefore

$$\psi_D(x) \leq M_3 + \frac{1}{2\pi}\|\zeta_0\|_1 \log\frac{2}{|x|}, \tag{3.12}$$

for every x such that $|x| > 1+2\sqrt{2}l$. From (3.12) we infer the existence of a positive constant N such that for $|x| > N$, $\psi_D(x) < \gamma$. We set $D_* := [-N,N]\times[-N,N]$, and define

$$W := \{\zeta|\ \zeta \in \Sigma(I,\leq,D) \text{ for some } D \supseteq D_*, \zeta \text{ is } DSS, \Im(\zeta) < I\}.$$

Note that to prove the lemma it suffices to show that $W = \emptyset$. Let us suppose the contrary. Then, without loss of generality, we may assume that there exist sequences $\{D_n\}_{n=1}^{\infty}$ and $\{\zeta_n\}_{n=1}^{\infty}$ such that $\zeta_n \in \Sigma(I,\leq,D_n)$ and $\Im(\zeta_n) < I$, for all $n \in \mathbb{N}$. Moreover, the D_n's can be assumed to be increasing and nondegenerate; that is, $D_n \subset D_{n+1}$, for all $n \in \mathbb{N}$ and $\bigcup\{D_n \mid n \in \mathbb{N}\} = \mathbb{R}^2$. To derive a contradiction we fix $\overline{\zeta} \in W$. Hence there exists D such that $\zeta_D := \overline{\zeta} \in \Sigma(I,\leq,D)$. Since $\Im(\zeta_D) < I$, we can find a strong neighbourhood of ζ_D, say U, relative to $\mathcal{F}(D)$, such that for $\zeta \in U$, $\Im(\zeta) < I$. Thus ζ_D is a local maximiser of Ψ relative to $\mathcal{F}(D)$. Now we can apply Lemma 6 to deduce existence of an increasing function, ϕ_D, such that

$$\zeta_D = \phi_D \circ \psi_D, \text{ a.e. in } D. \tag{3.13}$$

From (3.13) it follows that $\mathrm{supp}(\zeta_D) \subseteq D_*$.

In order to be able to apply Lemma 11 we need use the *modification process*, that is, to modify the *profile* function ϕ_D, so that an analogous equation to (3.13) holds almost everywhere in $\mathbb{R}^2$. We do this as follows. Since $\psi_D \in C^1(\mathbb{R}^2)$, we set $\alpha := Min_{x\in D}\ \psi_D(x)$. Note that this minimum is attained on ∂D. By (3.12) we deduce that there exists $n_0 \in \mathbb{N}$ such that $D_{n_0} \supset D$ and $\psi_D(x) < \frac{1}{2}\alpha$, for all $x \in \mathbb{R}^2 \setminus D_{n_0}$. Since $\zeta_{n_0} \in \Sigma(I,\leq,D_{n_0})$ and $\Im(\zeta_{n_0}) < I$, it follows that ζ_{n_0} is a local maximiser of Ψ relative to $\mathcal{F}(D_{n_0})$, hence there exists an increasing function ϕ_{n_0} such that $\zeta_{n_0} = \phi_{n_0}\circ\psi_{n_0}$, almost everywhere in D_{n_0}, hence it follows that $\mathrm{supp}(\zeta_{n_0}) \subseteq D_*$. Moreover, since ζ_D maximises Ψ relative to $\mathcal{F}(D_*)$ we obtain $\Psi(\zeta_{n_0}) \leq \Psi(\zeta_D)$ and this, in turn, implies that ζ_D is a local maximiser of Ψ relative to $\mathcal{F}(D_{n_0})$. Thus there exists an increasing function $\overline{\phi}_{D_{n_0}}$ such that

$$\zeta_D = \overline{\phi}_{D_{n_0}} \circ \psi_D, \text{ a.e. in } D_{n_0}.$$

Now we define the desired profile function, $\phi^\dagger$, as follows

$$\phi^\dagger(t) = \begin{cases} \overline{\phi}_{D_{n_0}}(t) & t \geq \alpha \\ 0 & t < \alpha \end{cases}$$

Then $\zeta_D = \phi^\dagger \circ \psi_{n_0}$, almost everywhere in $\mathbb{R}^2$. From $-\Delta\psi_D = \zeta_D$ almost everywhere in $\mathbb{R}^2$ we infer

$$-\Delta\psi_D = \phi^\dagger \circ \psi_D, \text{ a.e. in } \mathbb{R}^2.$$

Therefore ψ_D is a distributional solution, see [1] for the definition, of the following semilinear elliptic partial differential equation

$$-\Delta u = \phi^\dagger \circ u.$$

From Lemma 11 we deduce that $\psi_D = \psi_D^*$, since ψ_D has admissible asymptotic behaviour, by [24, Appendix A]. Hence ζ_D is also a spherically symmetric decreasing function, hence $\zeta_D = \zeta_0^*$. Thus $\Im(\zeta_D) = \Im(\zeta_0^*)$, whence $I > \Im(\zeta_0^*)$, which is a contradiction. This completes the proof. $\diamondsuit$

Remark. Note that a close inspection of the proof shows that D_* depends only on D_1. In fact we conclude that if we have an interval of the form $[a, \Im(\zeta_0^*))$ with $a > 0$, then there exists a square D_* such that for all $D \supseteq D_*$ and for all $I \in [a, \Im(\zeta_0^*))$, $\Sigma(I, D) \neq \emptyset$.

We now intend to derive the Euler-Lagrange equation satisfied by members of $\Sigma(I, D)$, for sufficiently large D and appropriate I. To do this we need to introduce the *value function.* Let $D \supseteq D_0$ and define $V_D : \Im(\mathcal{F}(D)) \to \mathbb{R}$ by

$$V_D(J) = \sup_{\zeta \in \mathcal{F}(D), \Im(\zeta) = J} \Psi(\zeta). \tag{3.14}$$

Remark. From Lemma 5(iv) and the fact that Ψ is weakly sequentially continuous on $L^p(D)$ we deduce that

$$V_D(J) = \sup_{\zeta \in \overline{\mathcal{F}(D)}^w, \Im(\zeta) = J} \Psi(\zeta).$$

Moreover, if $D_1 \supseteq D_0$, then by Lemma 12 and the remark following it there exists $D_* \supseteq D_1$ such that for all $D \supseteq D_*$, V_D is non-decreasing on $\text{int}(I_{D_1})$. To see this assume I_1, I_2 are in $\text{int}(I_{D_1})$, say $I_1 < I_2$. Then from Lemma 12 we have

$$V_D(I_j) = \sup_{\zeta \in \mathcal{F}(D), \Im(\zeta) \leq I_j} \Psi(\zeta),$$

j=1, 2. Hence $V_D(I_1) \leq V_D(I_2)$; that is V_D is increasing on $\text{int}(I_{D_1})$ for D and D_1 as in the preceding remark.

Next we claim that V_D is locally Lipschitz on $\text{int}(\Im(\overline{\mathcal{F}(D)^w}))$; the truth of this claim emerges from a technical lemma which is to follow, but first we recall the definition of the *Hausdorff metric*. If A, B are nonempty closed bounded sets in a metric space (X,d) we set $h(A,B) := \sup\{d(a,B) | \, a \in A\}$, where $d(a,B)$ denotes the usual distance from a to B, and define the Hausdorff metric $d_H(A,B) := max\{h(A,B),\ h(B,A)\}$. Next, we fix $D \supseteq D_0$ and define $C(I) = \overline{\mathcal{F}(D)^w} \cap \Im^{-1}(I)$, for $I > 0$. We now prove

Lemma 13 *$C(\cdot)$ is locally Lipschitz on $int(\Im(\overline{\mathcal{F}(D)^w}))$.*

Proof. Let $\Im(\overline{\mathcal{F}(D)^w}) := [c,d]$ and suppose $[a,b] \subset (c,d)$. Let $I, J \in [a,b]$, say $J < J$. Let us fix $\zeta_1 \in \overline{\mathcal{F}(D)^w} \cap \Im^{-1}(d) := C(d)$. Clearly there is $\lambda \in (0,1)$ such that $I = (1-\lambda)J + \lambda d$, indeed $\lambda = (I-J)/(d-J)$. For $\zeta_J \in C(J)$ we have $\Im(\lambda\zeta_1 + (1-\lambda)\zeta_J) = \lambda\Im(\zeta_1) + (1-\lambda)\Im(\zeta_J) = \lambda d + (1-\lambda)J = J$. Thus $\lambda\zeta_1 + (1-\lambda)\zeta_J \in C(I)$. Whence $\lambda\zeta_1 + (1-\lambda)C(J) \subseteq C(J)$, or equivalently $C(J) \subseteq (-\lambda)/(1-\lambda)\zeta_1 + 1/(1-\lambda)C(I)$. Since $C(I)$ is convex we may write $C(I) = \lambda C(I) + (1-\lambda)C(I)$. Therefore

$$\begin{aligned} C(J) &\subseteq (-\lambda)/(1-\lambda)\zeta_1 + \lambda/(1-\lambda)C(I) + C(I) \\ &\subseteq 2\lambda/(1-\lambda)\|\zeta_0\|_p \, B_1 + C(I) \\ &\subseteq 2(I-J)/(d-b)\|\zeta_0\|_p \, B_1 + C(I). \end{aligned} \tag{3.15}$$

Similarly one can show

$$C(I) \subseteq 2(I-J)/(a-c)\|\zeta_0\|_p \, B_1 + C(J). \tag{3.16}$$

It now follows from (3.15) and (3.16) that $C(\cdot)$ is Lipschitz on $[a,b]$. Hence we are done.$\diamondsuit$

We now prove

Lemma 14 *Let $D \supseteq D_0$, then $V := V_D$ is locally Lipschitz on $int(\Im(\overline{\mathcal{F}(D)^w}))$.*

Proof. To this end, let $[a,b] \subset \text{int}(\Im(\overline{\mathcal{F}(D)^w}))$ and consider $I, J \in [a,b]$. Fix $\epsilon > 0$ and let us assume that $V(I) \geq V(J)$. Then there exists $\zeta_1 \in C(I)$ such that

$$V(I) - V(J) \leq \Psi(\zeta_1) - \Psi(\zeta),\ \ \forall \zeta \in C(J).$$

Hence, since Ψ is locally Lipschitz on $\overline{\mathcal{F}(D)^w}$, we obtain

$$V(I) - V(J) \leq k\|\zeta_1 - \zeta\|_p, \ \ \forall \zeta \in C(J).$$

From the definition of $d(\zeta_1, C(j))$ we infer existence of $\zeta_2 \in C(J)$ such that

$$\|\zeta_1 - \zeta_2\|_p \leq d(\zeta_1, C(J)) + \epsilon.$$

Therefore,

$$V(I) - V(J) \leq k(d(\zeta_1, C(J)) + \epsilon) \leq k(d_H(C(I), C(J)) + \epsilon).$$

A similar inequality holds if $V(I) \leq V(J)$, so in general we obtain

$$|V(I) - V(J)| \leq k(d_H(C(I), C(J)) + \epsilon).$$

Finally, applying Lemma 13, and the fact that ϵ was arbitrary, we deduce that V is locally Lipshitz on $\overline{\mathcal{F}(D)^w}$.$\diamondsuit$

Lemma 15 *Let D_1 and D_* be as in Lemma 12. Assume $I \in int(I_{D_1})$ and let $\hat{\zeta} \in \Sigma(I, D)$, where $D \supseteq D_*$. Then we have*

$$\hat{\zeta} = \phi \circ (K\hat{\zeta} - \lambda x_2^2), \ \textit{a.e. in} \ \ D, \tag{3.17}$$

where ϕ is an increasing function and $\lambda \geq 0$, both unknown a priori.

Proof. For simplicity we set $V := V_D$. Now according to the definition of V it is clear that

$$V \circ \Im(\zeta) - \Psi(\zeta) \geq 0, \ \forall \zeta \in \overline{\mathcal{F}(D)^w}.$$

Note that for every $\zeta \in \Sigma(I, D)$, we have $V \circ \Im(\zeta) - \Psi(\zeta) = 0$; that is, all members of $\Sigma(I, D)$ are minimisers of $V \circ \Im(\zeta) - \Psi(\zeta)$ relative to $\overline{\mathcal{F}(D)^w}$. In particular, since $\hat{\zeta} \in \Sigma(I, D)$, $\hat{\zeta}$ minimises $V \circ \Im(\zeta) - \Psi(\zeta)$ on $\overline{\mathcal{F}(D)^w}$. On the other hand, from Lemma 3(vi) and Lemma 14 we infer that $V \circ \Im(\zeta) - \Psi(\zeta)$ is locally Lipschitz on a neighbourhood of $\hat{\zeta}$. Hence by a standard result from convex analysis we obtain

$$0 \in \partial^*(V \circ \Im - \Psi)(\hat{\zeta}) + N_{\overline{\mathcal{F}(D)^w}}(\hat{\zeta}),$$

where ∂^* denotes the generalised gradient and $N_{\overline{\mathcal{F}(D)^w}}(\hat{\zeta})$ stands for the normal cone to $\overline{\mathcal{F}(D)^w}$ at $\hat{\zeta}$. Thus there exist $\lambda \in \partial^* V(\Im(\hat{\zeta}))$ and $\xi \in L^q(D)$ for which

$$\lambda\Im(\zeta) - \Psi'(\hat{\zeta})(\zeta) + \xi(\zeta) = 0, \ \forall\zeta \in L^p(D)$$

Since V is non-decreasing on $\mathrm{int}(I_{D_1})$, it follows that $\lambda \geq 0$. From this we obtain,

$$\lambda\Im(\zeta - \hat{\zeta}) - \Psi'(\hat{\zeta})(\zeta - \hat{\zeta}) + \xi(\zeta - \hat{\zeta}) = 0, \ \forall\zeta \in L^p(D).$$

If, in particular, $\zeta \in \overline{\mathcal{F}(D)^w}$, then it follows that

$$\Psi'(\hat{\zeta})(\zeta) - \lambda\Im(\zeta) \leq \Psi'(\hat{\zeta})(\hat{\zeta}) - \lambda\Im(\hat{\zeta}).$$

Therefore $\hat{\zeta}$ maximises $\Psi'(\hat{\zeta})(\zeta) - \lambda\Im(\zeta)$ relative to $\overline{\mathcal{F}(D)^w}$. In other words, $\hat{\zeta}$ maximises $\int \zeta(K\hat{\zeta} - \lambda x_2^2)$ relative to $\zeta \in \overline{\mathcal{F}(D)^w}$, where the integral is over $\mathbb{R}^2$. Whence, by [Chapter 2, Lemma 4], there exists an increasing function ϕ such that $\hat{\zeta} = \phi \circ (K\hat{\zeta} - \lambda x_2^2)$, almost everywhere in D. Hence we derive (3.17). This completes the proof. $\diamondsuit$

Remark. Lemmas 12 and 15 imply that for every $0 < I < \Im(\zeta_0^*)$, there exists $D(I)$, a square, such that for all $D \supseteq D(I)$ then $\Sigma(I, D)$ contains DSS members, and if $\zeta \in \Sigma(I, D)$, then $\zeta = \phi \circ (K\zeta - \lambda x_2^2)$, almost everywhere in D, for some non-decreasing function ϕ and $\lambda \geq 0$.

We now present the following definitions.

Definition. For $I \geq 0$ and a square D we define

$$M(I, D) := \{(\zeta, \phi, \lambda) | \, \zeta \in \Sigma(I, D), \text{ and } \zeta = \phi \circ (K\zeta - \lambda x_2^2), \text{ a.e. in } D\}$$

Definition. For $0 < I < \Im(\zeta_0^*)$ and $D(I)$ as in the preceeding remark we define

$$\Lambda(I) = \{\lambda | \, (\zeta, \phi, \lambda) \in M(I, D) \text{ for some } D \supseteq D(I), \ \zeta \in L^p(D) \text{ and } \phi\}.$$

Lemma 16 *Let $0 < I < \Im(\zeta_0^*)$, then $\Lambda(I)$ is bounded from above.*

Proof. Assume the assertion of the lemma is not true. Then there exists a sequence of non-negative numbers $\{\lambda_n\}_{n=1}^{\infty}$ in $\Lambda(I)$ such that $\lambda_n \to \infty$ as $n \to \infty$. This, in turn, implies the existence of sequences $\{D_n\}_{n=1}^{\infty}$, $D_n \supseteq D(I)$, for all $n \in \mathbb{N}$, and $\{\zeta_n\}_{n=1}^{\infty}$ such that $\zeta_n \in \Sigma(I, D_n)$, for all $n \in \mathbb{N}$, and a sequence of increasing functions, $\{\phi_n\}_{n=1}^{\infty}$ for which the following holds

$$\zeta_n = \phi_n \circ (\psi_n - \lambda_n x_2^2), \text{ a.e. in } D_n, \ \forall n \in \mathbb{N}. \tag{3.18}$$

In addition, according to the last remark, we may assume that the ζ_n are DSS. For $\alpha > 0$, we denote the strip $\mathbb{R} \times (-\alpha, \alpha)$ by S_α. For $\alpha_0 := (I/2\|\zeta_0\|_1)^{1/2}$ we claim that

$$|\text{supp}(\zeta_n) \cap (\mathbb{R}^2 \setminus S(\alpha_0))| > 0, \ \forall n \in \mathbb{N}.$$

To see this assume otherwise. Then for some n_0 we would have $\text{supp}(\zeta_{n_0}) \subseteq S(\alpha_0)$, whence

$$\Im(\zeta_{n_0}) \leq {\alpha_0}^2 \|\zeta_0\|_1 = I/2,$$

which is a contadiction proving our claim.

Next we fix $R > 0$ sufficiently large to ensure $|B_R \cap S(\alpha_0)| > 2\pi a^2$. Therefore, since the ζ_n are DSS, we deduce

$$|\text{supp}(\zeta_n) \cap (\mathbb{R}^2 \setminus S(\alpha_0)) \cap B_R| > 0, \ \forall n \in \mathbb{N}. \tag{3.19}$$

Without loss of generality, we may assume that $D_n \supseteq B_R$, for all $n \in \mathbb{N}$. Now let us consider the points $x = (x_1, x_2)$, $y = (y_1, y_2)$ in B_R such that $x_2 > \alpha_0$ and $0 < y_2 < \alpha_0/2$. Applying the Mean Value Inequality and estimate (3.9) yields

$$\psi_n(y) \geq \psi_n(x) - C|x - y| \geq \psi_n(x) - 2CR.$$

Therefore we obtain

$$\psi_n(y) - \lambda_n y_2^2 \geq -2CR + \psi_n(x) - \lambda_n x_2^2 + (\lambda_n x_2^2 - \lambda_n y_2^2).$$

Since $\lambda_n \geq 0$ and $y_2 < \alpha_0/2 < x_2/2$ we deduce

$$\psi_n(y) - \lambda_n y_2^2 \geq \psi_n(x) - \lambda_n x_2^2 + 3/4\lambda_n {\alpha_0}^2 - 2CR.$$

Now let n_* be sufficiently large to ensure $3/4\lambda_{n_*}{\alpha_0}^2 - 2CR > 0$, which is possible since $\lambda_n \to \infty$ as $n \to \infty$. Then we find that

$$\psi_{n_*}(y) - \lambda_{n_*} y_2^2 > \psi_{n_*}(x) - \lambda_{n_*} x_2^2 \tag{3.20}$$

On the other hand, $\text{supp}(\zeta_{n_0}) \cap B_R$ intersects both $S(\alpha_0)$ and $\mathbb{R}^2 \setminus S(\alpha_0)$ in sets of positive measure, by (3.19). To derive a contradiction, let us consider (3.20) at some $x \in \text{supp}(\zeta_{n_*})$ and $y \in \mathbb{R}^2 \setminus \text{supp}(\zeta_{n_*})$. We may assume that the intersection of every ball centered at either x or y, with $\text{supp}(\zeta_{n_*})$ has positive measure. It then follows, from the continuity of $\psi_{n_*}(x) - \lambda_{n_*} x_2^2$ and (3.20), that there are two sets of positive measure, say A and E, in D_{n_*} such that $A \subset \text{supp}(\psi_{n_*})$, $E \cap \text{supp}(\psi_{n_*}) = \emptyset$, for which the following holds

$$\psi_{n_*}(v) - \lambda_{n_*}v_2^2 > \psi_{n_*}(w) - \lambda_{n_*}w_2^2, \ \forall v \in E, \ \forall w \in A.$$

However, this contradicts (3.18) for $n = n_*$. Therefore $\Lambda(i)$ is bounded from above. $\diamondsuit$

We are now in the position to prove the theorem.

Proof of the theorem. we break the proof into three cases.

Case 1: $0 < I < \Im(\zeta_0^*)$. Let $D \supseteq D(I)$ introduced prior to Lemma 16, and assume $\zeta_D \in \Sigma(I, D)$ which is DSS. By Lemma 15, there exists an increasing function ϕ_D and $\lambda_D \geq 0$ for which we have

$$\zeta_D = \phi_D \circ (\psi_D - \lambda_D x_2^2), \text{ a.e. in } D, \tag{3.21}$$

where $\psi_D := K\zeta_D$. For $\zeta_i \in \Sigma(I, D(I))$ we have

$$\Psi(\zeta_i) \leq \Psi(\zeta_D) := \frac{1}{2}\int_{\mathbb{R}^2} \zeta_D \psi_D \leq \frac{1}{2}||\zeta_0||_1 \psi_D(0),$$

since ψ_D is DSS. Thus if we set $M_1 = 2||\zeta_0||_1^{-1}\Psi(\zeta_I)$, then $\psi_D(0) \geq M_1$. Let λ_* denote an upper bound for $\Lambda(I)$, and let $D^I := [-T, T] \times [-T, T]$, then for $x \in D^I$ we obtain

$$|\nabla(\psi_D(x) - \lambda_D x_2^2)| \leq M_2||\zeta_D||_p + 2\lambda_D|x_2| \leq C + 2\lambda_* T := M_4,$$

where we have used (3.9). Now an application of the Mean Value Inequality and the estimate $\psi(0) \geq M_1$ yields

$$\psi_D(x) - \lambda_D x_2^2 \geq \psi_D(0) - \sqrt{2}M_4T \geq M_1 - \sqrt{2}M_4T := \gamma, \ \forall x \in D^I.$$

Therefore, we conclude that

$$\inf_{D \supseteq D^I} \inf_{x \in D^I} (\psi_D(x) - \lambda_D x_2^2) \geq \gamma. \tag{3.22}$$

Now, by (3.12), there exists $R > 0$ independent of D such that for $|x| > R$, $\psi_D(x) < \gamma$. Hence, $\psi_D(x) - \lambda_D x_2^2 < \gamma$, whenever $|x| > R$, since $\lambda_D \geq 0$. If we set $D^* := [-R, R] \times [-R, R]$, then

$$\psi_D(x) - \lambda_D x_2^2 < \gamma, \ \forall x \in \mathbb{R}^2 \setminus D^*. \tag{3.23}$$

Note that (3.21) implies $\psi_D(x) - \lambda_D x_2^2$ will attain its larger values on $\text{supp}(\zeta_D)$, by

[Chapter 2, Lemma 7]. Therefore in view of (3.22) and (3.23), we must have

$$\mathrm{supp}(\zeta_D) \subseteq D^*, \ \forall D \supset D^I \cup D^*.$$

Thus, if we set $\hat{D} := D^* \cup D^I$, then $\zeta_{\hat{D}}$, a DSS member of $\Sigma(I, \hat{D})$, will be a solution to $P(I)$. It remains to derive (3.4), thus we consider $\zeta \in \Sigma(I)$. Then there exists D such that $\mathrm{supp}(\zeta) \subset D$; moreover, the following holds

$$\zeta = \phi \circ (\psi - \lambda x_2^2), \text{ a.e. in } D,$$

where ϕ is an increasing function, $\lambda \geq 0$ and $\psi := K\zeta$. Clearly to obtain (3.4) we need to modify ϕ. This is done as follows. Since $\psi - \lambda x_2^2 \in C^1(\mathbb{R}^2)$, we may define $\beta := Min_{x \in D} \psi(x) - \lambda x_2^2$. Once again using (3.12), we can find a square $D' \supseteq D$ such that $\psi(x) - \lambda x_2^2 < \beta$, $\forall x \in \mathbb{R}^2 \setminus D'$. Since $\zeta \in \Sigma(I, D')$, we also obtain

$$\zeta = \phi' \circ (\psi - \lambda' x_2^2), \text{ a.e. in } D',$$

where ϕ' is an increasing function and $\lambda' \geq 0$. Now we define $\hat{\phi}$ as follows

$$\hat{\phi}(t) = \begin{cases} \phi'(t) & t \geq \beta \\ 0 & t < \beta. \end{cases}$$

Therefore, we obtain

$$\zeta = \hat{\phi} \circ (\psi - \lambda' x_2^2), \text{ a.e. in } \mathbb{R}^2.$$

By Lemma 2(i), $\zeta = -\Delta K\zeta$, almost everywhere in $\mathbb{R}^2$, hence (3.4).
case 2: $I > \Im(\zeta_0^*)$. As it was pointed out earlier, the existence part of the theorem becomes trivial in this case. In fact, if $I > \Im(\zeta_0^*)$, $(\zeta_0^*)^t$ will be a solution, for appropriate $t \in \mathbb{R}$. To derive (3.4) we again use the value function. To be specific, let $\zeta \in \Sigma(I)$, then there exists D such that $\mathrm{supp}(\zeta) \subset D$ and $I \in \mathrm{int}\Im(\overline{\mathcal{F}(D)^w})$. Now we set $V := V_D$, as defined in (3.14). Note that $V(J) = \Psi(\zeta_0^*)$, when J is close to I and D is sufficiently large. Hence, V, being constant near I, will be locally Lipschitz near I. Thus, exactly as in case 1, we obtain $\zeta = \phi(K\zeta)$ a.e. in D. We remark that in this case , since V is constant near I, the Lagrange multiplier, λ, is equal to zero. Finally, by a modification process we construct $\hat{\phi}$ such that $\zeta = \hat{\phi}(K\zeta)$ a.e. in $\mathbb{R}^2$. Again refering to Lemma 1(i), we deduce (3.4).
case 3: $I = \Im(\zeta_0^*)$. Obviously, ζ_0^* is a solution to $P(I)$. It then remains to derive (3.4); let $\zeta \in \Sigma(I)$. Then some translation ζ^t of ζ belongs to $\Sigma(J)$ for some $J > I$, so by case

2 we have

$$\zeta^t = \phi \circ K\zeta^t, \text{ a.e. in } \mathbb{R}^2,$$

for some increasing function ϕ. This, in turn, implies

$$\zeta = \phi \circ K\zeta, \text{ a.e. in } \mathbb{R}^2.$$

This completes the proof of the theorem. $\Diamond$

Chapter 4

An obstacle problem

4.1 Introduction

This chapter complements the preceding chapters. Here, we prove existence of a two dimensional steady flow of an ideal fluid past an obstacle, containing a bounded vortex anomaly and approaching a shear flow at infinity; moreover, the flow is tangential to the boundary of the domain of interest, Ω, which is an unbounded exterior region.

The variational formulation of the problem is similar to that in chapter one; however, there are two main differences, both caused by the presence of the obstacle. The first of these is that we no longer have an explicit formula for the Green's function for $-\Delta$ with homogeneous Dirichlet boundary conditions , in Ω. To overcome this difficulty we estimate the Green's function with the ones having known formulae. The second difference is the lack of symmetry of Ω; this makes Steiner-symmetrisation, for instance, unavailable. However, as it will be shown, prescribing a suitable circulation around the boundary, $\partial\Omega$, will enable us to get around this issue.

Finally, to deal with the lack of compactness, again due to the unboundedness of Ω, we will follow [6], that is, we first truncate the domain and implement Burton's theory to prove solvability of the corresponding variational problem, and then show that maximisers remain the same for sufficiently large bounded domains.

4.2 Notation, definitions and statement of results

In this section we will state the main theorem of this chapter followed by the strategy to be used.

4.2.1 Notation and definitions

Let D be an open, bounded and smooth($\partial D \in C^{2,\gamma}$) set in $\mathbb{R}^2$ which contains the origin in its interior. The domain of interest is $\Omega := \mathbb{R}^2 \setminus \overline{D}$. The ball centered at x with radius r is denoted by $B_r(x)$ and B_r indicates the ball centered at the origin with radius r. We assume that $\overline{D} \subset B_1$ and fix $\overline{B_R} \subset D$, for some $R > 0$. Truncations of Ω are defined by $\Omega_n := \Omega \cap B_n$ for all $n \geq 1$. x, y, etc. denote generic points in $\mathbb{R}^2$ as in previous chapters. The symbol G will always denote the Green's function for $-\Delta$ with homogeneous Dirichlet boundary conditions on some domain. In particular, G, G_R and G_n denote the Green's functions on Ω, $\mathbb{R}^2 \setminus \overline{B_R}$ and Ω_n, respectively. The existence of G will be addressed in the next section; on the other hand it is a standard result that

$$G_R(x,y) = \frac{1}{2\pi} \log \frac{R^{-1}|x||y-x^*|}{|y-x|}, \; x, y \in \mathbb{R}^2 \setminus \overline{B_R}, \; x \neq y,$$

where $x^* = R^2 x/|x|^2$, the inversion of x in the circle B_R, see for example [37]. If ζ is a measurable function on Ω we define

$$K\zeta(x) := \int_\Omega G(x,y)\zeta(y)dy,$$

for all x in $\mathbb{R}^2$ for which the integral exists. Accordingly, we define

$$\begin{aligned} K_R\zeta(x) &:= \int_{\mathbb{R}^2\setminus\overline{B_R}} G_R(x,y)\zeta(y)dy \\ K_n\zeta(x) &:= \int_{\Omega_n} G_n(x,y)\zeta(y)dy, \end{aligned}$$

whenever the integrals exist. Next we define

$$\eta_c(x) := \eta_1(x) + x_2^2 + c\log|x|, \; x \in \Omega, \; c \in \mathbb{R} \tag{4.1}$$

where $\eta_1 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies

$$\begin{cases} \Delta\eta_1 &= 0 \text{ in } \Omega \\ \eta_1 &= -x_2^2 - c\log|x| \text{ on } \partial\Omega \\ \nabla\eta_1 &\to 0 \text{ as } |x| \to \infty. \end{cases}$$

The existence of η_1 will be demonstrated in section 4.3.

For a measurable function ζ on Ω we define

$$\Psi_1(\zeta) := \int_\Omega \zeta K \zeta$$
$$\Im(\zeta) := \int_\Omega \zeta \eta_c,$$

whenever the integrals exist; accordingly the *energy functional* Ψ is defined as follows

$$\Psi(\zeta) := \Psi_1(\zeta) - \Im(\zeta).$$

Definition. If $\psi \in C^1(\overline{\Omega})$ represents the stream function of a flow, then

$$Cir(\psi, \partial\Omega) := \int_{\partial\Omega} \partial_{\vec{n}} \psi d\sigma,$$

where $\vec{n}$ denotes the exterior unit normal vector to $\partial\Omega$, is called *the circulation* of the flow.

Definition. Suppose $\psi \in C^1(\overline{\Omega})$ represents the stream function of a flow. We say the flow is *tangential* to $\partial\Omega$ if the normal component of the corresponding velocity field vanishes on $\partial\Omega$, that is,

$$\nabla^\perp \psi \cdot \vec{n} = 0.$$

For a real number p we denote by p^* the conjugate exponent of p. Similarly to previous chapters it is always assumed that $p > 2$. For a measurable function ζ on Ω, $\mathrm{supp}(\zeta)$ denotes *the strong support* of ζ and is defined by

$$\mathrm{supp}(\zeta) := \{x \in \Omega | \ \zeta(x) > 0\}.$$

Let $\zeta_0 \in L^p(\Omega)$ be a non-zero, non-negative function vanishing outside a bounded set such that $||\zeta||_1 = 1$ and $|\mathrm{supp}(\zeta_0)| = \pi a^2$ for some $a > 0$. The set of rearrangements of ζ_0, on Ω, which vanish outside bounded sets are denoted by $\mathcal{F}$. If S is a measurable subset of Ω such that $|S| > \pi a^2$ then the subset of $\mathcal{F}$ comprising functions vanishing outside S is denoted by $\mathcal{F}(S)$. Accordingly $\overline{\mathcal{F}(S)}^w$ denotes the weak closure of $\mathcal{F}(S)$ in $L^p(S)$. Thus from [11] it follows that $\overline{\mathcal{F}(S)}^w$ is weakly sequentially compact and convex in $L^p(S)$.

4.2.2 Formulation of the problem

We will prove that the flow in question can be represented by an extremal of a variational problem. Specifically, (P) denotes the following maximisation problem

$$(P): \sup_{\zeta \in \mathcal{F}} \Psi(\zeta)$$

and Σ denotes the set of solutions of (P). Then we prove the following

Theorem 1 *For $c \geq 3$, problem (P) has a solution, that is, $\Sigma \neq \emptyset$. Moreover, if $\zeta \in \Sigma$ then $\psi := K\zeta$ satisfies*

(i) $\psi \in C^1(\overline{\Omega})$ up to a set of measure zero

(ii) $\psi = 0$ on $\partial\Omega$

(iii) $-\Delta\psi = \phi \circ (\psi - \eta_c)$, almost everywhere in Ω, where ϕ is some increasing function which is unknown a priori.

As mentioned in the Introduction the main difficulty in proving the existence part of Theorem 1 is the lack of compactness which is caused by the unboundedness of Ω. Hence we can not apply the conventional direct method of calculus of variations. To overcome this difficulty we again follow the approach proposed by [6]. We will first consider (P) on a bounded domain; specifically, we introduce the problem

$$(P_n): \sup_{\zeta \in \mathcal{F}(\Omega_n)} \Psi(\zeta),\ n \in \mathbb{N}.$$

Accordingly we let Σ_n denote the set of solutions of (P_n). To ensure $\mathcal{F}(\Omega_n) \neq \emptyset$ we impose $n > (1+a^2)^{1/2}$. We will see, by applying Burton's theory, that $\Sigma_n \neq \emptyset$. We then prove the existence of a critical number, n^*, such that for any $n \geq n^*$ and $\zeta \in \Sigma_n$ we have

$$\operatorname{supp}(\zeta) \subset \Omega_{n^*}.$$

Therefore, any $\zeta_{n^*} \in \Sigma_{n^*}$ will be a member of Σ. Hence, $\Sigma \neq \emptyset$.

4.3 Preliminary results

In this section we will first address the issue of existence of G. This seems to be a widely accepted result but we have not been able to find a precise reference. Having proved the well-definedness of G we proceed to define the integral operator K and state some properties of $K\zeta$, including interior and global regularity, when $\zeta \in L^p(\Omega)$ vanishes outside a bounded set. We will then prove the existence of η_c. Finally, this section will be cocluded by stating a result from Burton's Theory.

4.3.1 Existence of the Green's function

The existence of G is an immediate consequence of the following lemma; but first we introduce $\hat{G}_n$ to be the zero-extension of G_n, that is,

$$\hat{G}_n(x,y) := \begin{cases} G_n(x,y) & x,y \in \Omega_n,\ x \neq y \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1 *For every* $(x,y) \in \Omega \times \Omega$,

$$\lim_{n\to\infty} \hat{G}_n(x,y)$$

exists.

Proof Clearly it suffices to show that $\{\hat{G}_n\}_{n=1}^{\infty}$ is an increasing and bounded sequence on $\Omega \times \Omega$. To prove this sequence is increasing it is enough to show

$$G_{n+1}(x,y) \geq G_n(x,y),\ \ x,y \in \Omega_n,\ x \neq y,\ n \in \mathbb{N}. \tag{4.2}$$

Recall that we can write

$$G_n(x,y) = \frac{1}{2\pi}\log\frac{1}{|x-y|} - h_n(x,y),\ n \in \mathbb{N},$$

where h_n is the harmonic part of G_n. Therefore, the truth of (4.2) will emerge once we show

$$h_{n+1}(x,y) \leq h_n(x,y),\ \ x,y \in \Omega_n,\ n \in \mathbb{N}.$$

For this purpose we fix $y \in \Omega_n$. Since $G_{n+1}(x,y) \geq 0$ for $x \in \Omega_{n+1} \setminus \{y\}$, it follows that

$$h_{n+1}(x,y) \leq 1/(2\pi)\log 1/|x-y|,\ \ x \in \Omega_{n+1} \setminus \{y\}.$$

In particular, $h_{n+1}(x,y) \leq 1/(2\pi)\log 1/|x-y|$ for all $x \in \partial B_n$. Next we set $u_n(x) := h_n(x,y) - h_{n+1}(x,y)$. Therefore, u_n is clearly a classical solution of the following Dirichlet problem:

$$\begin{cases} \Delta w &= \quad 0 \quad \text{in } \Omega_n \\ w &= h_n(\cdot,y) - h_{n+1}(\cdot,y) \ \text{ on } \partial\Omega_n. \end{cases}$$

Observe that u_n vanishes on $\partial\Omega$ and is non-negative on ∂B_n, hence u_n is non-negative on $\partial\Omega_n$. Whence, by the Maximum Principle, u_n is non-negative in $\overline{\Omega}_n$. This implies $h_n(x,y) \geq h_{n+1}(x,y)$ for all $x \in \Omega_n$, as desired.To prove boundedness, first we note that similar arguments as above leads to

$$G_n(x,y) \le G_R(x,y), \;\; x,y \in \Omega_n, \; n \in \mathbb{N},$$

where R is the radius of a ball $B_R \subset D$; such a ball exists since 0 is an interior point of D. Denoting the zero-extension of G_R by $\hat{G}_R$ we easily obtain $\hat{G}_n(x,y) \le \hat{G}_R(x,y)$ for all $x,y \in \Omega$ and all $n \in \mathbb{N}$. Hence we are done.◇

Lemma 2 *The Green's function for $-\Delta$, on Ω, with homogeneous Dirichlet boundary conditions exists and is denoted by G.*

Proof This is a consequence of Lemma 1 and [17, Proposition 12, p364].

4.3.2 Properties of the operator K

In the results to follow we make use of the following elementary estimate:

$$(\dagger) \quad G_R(x,y) \le \begin{cases} 1/(2\pi)\log 4R^{-1}|x|, & |y| \ge 2|x|, \\ 1/(2\pi)\log(2R^{-1}|x|^2+1)/|y-x|, & |y| < 2|x|. \end{cases}$$

An immediate consequence of (†) is that $K_R\zeta(x)$ is well defined for all $x \in \mathbb{R}^2$, provided $\zeta \in L^p(\mathbb{R}^2 \setminus \overline{B_R})$ vanishes outside a bounded set. To see this we proceed as follows: For $x \in \mathbb{R}^2$ we have

$$\begin{aligned} |K_R\zeta(x)| &\le \int_{\mathbb{R}^2\setminus\overline{B_R}} G_R(x,y)|\zeta(y)|dy \\ &= \int_{(\mathbb{R}^2\setminus\overline{B_R})\cap\{|y|\ge 2|x|\}} G_R(x,y)|\zeta(y)|dy + \int_{(\mathbb{R}^2\setminus\overline{B_R})\cap\{|y|\le 2|x|\}} G_R(x,y)|\zeta(y)|dy \\ &= I_1 + I_2. \end{aligned}$$

An application of (†) implies

$$I_1 \le 1/(2\pi)\log(4R^{-1}|x|)\;\|\zeta\|_1 \le C_1\log(4R^{-1}|x|)\|\zeta\|_p, \tag{4.3}$$

and, again by (†),

$$\begin{aligned} I_2 &\le C_2\log(2R^{-1}|x|^2+1)\;\|\zeta\|_p \\ &+ 1/(2\pi)\int_{(\mathbb{R}^2\setminus\overline{B_R})\cap\{|y|\le 2|x|\}} \log\frac{1}{|y-x|}|\zeta(y)|dy. \end{aligned} \tag{4.4}$$

Now the integral in (4.4) can be estimated by an inequality due to Hardy. Let ζ^* denote the Schwarz-symmetrisation of the restriction of $|\zeta|$ to the set $(\mathbb{R}^2 \setminus \overline{B_R}) \cap \{|y| \le 2|x|\}$, with respect to x. Then, by [31], we obtain

$$\int_{(\mathbb{R}^2\setminus\overline{B_R})\cap\{|y|\le 2|x|\}} \log\frac{1}{|y-x|}|\zeta(y)|dy \le \int_{B_l(x)} \log\frac{1}{|y-x|}\zeta^*(y)dy$$

where l satisfies $\pi l^2 = |\mathrm{supp}(\zeta^*)|$. By an application of Hölder's inequality, we then obtain

$$\begin{aligned}\int_{B_l(x)} \log\frac{1}{|y-x|}\zeta^*(y)dy &\le \left(\int_{B_l(x)} |\log\frac{1}{|y-x|}|^q dy\right)^{1/q} ||\zeta||_p \\ &\le C_3||\zeta||_p,\end{aligned}$$

where C_3 depends only on l and p. Now from (4.4) we deduce

$$I_2 \le C_2\log(2R^{-1}|x|^2+1)\ ||\zeta||_p + C_3||\zeta||_p. \tag{4.5}$$

From (4.3) and (4.5) we infer

$$|K_R\zeta(x)| \le C(\log(4R^{-1}|x|) + \log(2R^{-1}|x|^2+1)+1)||\zeta||_p, \tag{4.6}$$

where C depends only on $\mathrm{supp}(\zeta)$ and p.

We now state the main result of this section.

Lemma 3 *Let U be an open and bounded subset of Ω. Then for any $q \ge 1$, $K : L^p(U) \longrightarrow L^q(U)$, is a linear compact operator in the sense that if $\{\zeta_n\}_{n=1}^\infty$ is a sequence of functions bounded in $L^p(\Omega)$ and vanishing outside U, then the sequence $\{K\zeta_n|_U\}_{n=1}^\infty$ has a subsequence converging in the L^q-norm. Moreover, if $\zeta \in L^p(\Omega)$ vanishes outside U, then*
(i) $-\Delta K\zeta = \zeta$ a.e. in Ω.
(ii) $K\zeta \in W^{2,p}(\Omega_n)$ for some $n \in \mathbb{N}$.
(iii) $\gamma(K\zeta, \partial\Omega, W^{1,p}(\Omega_{n_1})) = 0$, for some $n_1 \in \mathbb{N}$, in the sense that there exists a sequence of functions $\{\omega_n\}_{n=1}^\infty$ in $C^1(\Omega)$ such that $\omega_n \to K\zeta$ in $W^{1,p}(\Omega)$, and ω_n vanish near $\partial\Omega$. Hence $K\zeta$ vanishes on $\partial\Omega$, pointwise.

Proof. Fix $\zeta \in L^p(\Omega)$ which vanishes outside U. Let $n_0 \in \mathbb{N}$ such that $U \subset \Omega_{n_0}$. Set $n_1 := n_0 + 1$ and fix $n > n_1$. Let us denote by $\hat{K}_n\zeta$ the unique minimiser of the functional

$$F_n(u) := \frac{1}{2}\int_{\Omega_n} |\nabla u|^2 - \int_{\Omega_n} u\zeta,$$

relative to $u \in H_0^1(\Omega_n)$. In Appendix A it is proved that

$$\hat{K}_n\zeta = K_n\zeta, \text{ a.e. in } \Omega_n.$$

Therefore $K_n\zeta$ satisfies

$$\begin{cases} -\Delta K_n\zeta &= \zeta \text{ in } \mathcal{D}'(\Omega_n) \\ K_n\zeta &\in H_0^1(\Omega_n). \end{cases} \tag{4.7}$$

For a test function $\phi \in C_0^\infty(B_{n_1})$ where $\phi = 1$ on B_{n_0} we clearly have $\phi K_n\zeta \in H_0^1(\Omega_n)$. From (4.7) we infer

$$-\Delta(\phi K_n\zeta) = -\Delta\phi(K_n\zeta) - 2\nabla\phi\cdot\nabla K_n\zeta + \phi\zeta \quad \text{in } \mathcal{D}'(\Omega_{n_1}).$$

We set $f_n := -\Delta\phi(K_n\zeta) - 2\nabla\phi\cdot\nabla K_n\zeta + \phi\zeta$ and claim

$$\|f_n\|_{2,\Omega_{n_1}} \leq C(n_1),$$

where $C(n_1)$ depends only on n_1. To prove the claim it suffices to show

$$\|K_n\zeta\|_{2,\Omega_{n_1}} \leq C(n_1), \tag{4.8}$$

since $K_n\zeta$ being the minimiser of F_n yields $F_n(K_n\zeta) \leq F_n(0) = 0$, hence

$$\begin{aligned} \|\nabla K_n\zeta\|_{2,\Omega_{n_1}} \leq 2\int_{\Omega_n}\zeta K_n\zeta &= 2\int_{\Omega_{n_1}}\zeta K_n\zeta \\ &\leq 2\|\zeta\|_{2,\Omega_{n_1}}\|K_n\zeta\|_{2,\Omega_{n_1}}, \end{aligned}$$

where in the last inequality we have used the Hölder's inequality. Finally, we observe that (4.8) follows immediately from (4.6). Therefore $\phi K_n\zeta$ satisfies

$$\begin{cases} -\Delta(\phi K_n\zeta) &= f_n \text{ in } \mathcal{D}'(\Omega_{n_1}) \\ \phi K_n\zeta &\in H_0^1(\Omega_{n_1}). \end{cases}$$

Hence by applying [28, Theorem 8.12] we infer $\phi K_n\zeta \in W^{2,2}(\Omega_{n_1})$; moreover we have

$$\begin{aligned} \|\phi K_n\zeta\|_{W^{2,2}(\Omega_{n_1})} &\leq C(\|f_n\|_{2,\Omega_{n_1}} + \|\phi K_n\zeta\|_{2,\Omega_{n_1}}) \\ &\leq C(n_1). \end{aligned}$$

Therefore $\{\phi K_n\zeta\}_{n=1}^\infty$ is a bounded sequence in $W^{2,2}(\Omega_{n_1})$. Hence, $W^{2,2}(\Omega_{n_1})$ being a reflexive Banach space, it follows that $\{\phi K_n\zeta\}_{n=1}^\infty$ contains a weakly convergent subsequene, say $\{\phi K_{n_j}\zeta\}_{j=1}^{\infty}$. Thus there exists $\hat{u} \in W^{2,2}(\Omega_{n_1})$ such that

$$\phi K_{n_j}\zeta \rightharpoonup \hat{u} \quad \text{in } W^{2,2}(\Omega_{n_1}), \ \text{ as } j \to \infty.$$

Since $W^{2,2}(\Omega_{n_1})$ is compactly embedded in $C(\overline{\Omega}_{n_1})$, it follows that $\phi K_{n_j}\zeta \to \hat{u}$, uniformly in Ω_{n_1}. In particular, we infer $K_{n_j}\zeta \to \hat{u}$, pointwise in Ω_{n_0}. On the other hand an application of the Lebesgue Dominated Convergence Theorem yields $K_{n_j}\zeta \to K\zeta$ almost everywhere in Ω_{n_0}. Hence

$$\hat{u} = K\zeta \ \text{ a.e. in } \Omega_{n_0}.$$

Therefore by a diagonalisation process we deduce $K\zeta \in W^{2,2}_{loc}(\overline{\Omega})$ where

$$W^{2,2}_{loc}(\overline{\Omega}) := \{v \in W^{2,2}(\hat{\Omega}) |\ \hat{\Omega} \text{ is a bounded and open set such that } \overline{\hat{\Omega}} \subset \Omega\}.$$

Moreover, we obtain

$$\gamma(K\zeta, \partial\Omega, W^{1,2}(\Omega_{n_1})) = 0.$$

Now we intend to prove

$$-\Delta K\zeta = \zeta \ \text{ in } \mathcal{D}'(\Omega). \tag{4.9}$$

For this purpose we fix $\phi \in C_0^{\infty}(\Omega)$. Then by the Lebesgue Dominated Convergence Theorem we obtain

$$\int_{\Omega_m} K_m\zeta(-\Delta\phi) \to \int_{\Omega} K\zeta(-\Delta\phi) \ \text{ as } m \to \infty.$$

Also $\int_{\Omega_m} K_m\zeta(-\Delta\phi) = \int_{\Omega}\zeta\phi$, for every $m \in \mathbb{N}$. Hence we derive (4.9). The combination of $K\zeta \in W^{2,2}_{loc}(\overline{\Omega})$ and (4.9) yields (i). So far, we have been able to show, in particular, that

$$(*) \quad \begin{cases} K\zeta \in W^{2,2}(\Omega_{n_1}) \\ -\Delta K\zeta = \zeta \ \text{ a.e. in } \Omega_{n_1} \\ \gamma(K\zeta, \partial\Omega, W^{1,2}(\Omega_{n_1})) = 0. \end{cases}$$

To derive (ii) and (iii) we use a classical global regularity theory, namely [28, Theorem 9.16]. Indeed, from $(*)$ and [28, Theorem 9.16] we deduce $K\zeta \in W^{2,p}_{loc}(\Omega_{n_1} \cup \partial\Omega)$ and $\gamma(K\zeta, \partial\Omega, W^{1,p}(\Omega_{n_1})) = 0$; moreover, we have the following estimate

$$\|K\zeta\|_{W^{2,p}(U)} \leq C(\|\zeta\|_p + \|K\zeta\|_{p,\Omega_{n_1}}). \tag{4.10}$$

Hence from (4.10) and (4.6) it follows that $K : L^p(U) \longrightarrow W^{2,p}(U)$ is bounded. Since $W^{2,p}(U)$ is compactly embedded into $L^q(U)$, $q \geq 1$, we infer $K : L^p(U) \longrightarrow L^q(U)$ is compact. Let us point out that since $K\zeta \in W^{2,p}_{loc}(\Omega_{n_1} \cup \partial\Omega)$, we derive (ii) for $n = n_0$. To prove (iii) we use the Trace Theorem, see for example [1]. Since $K\zeta \in W^{2,p}(\Omega_{n_0})$ we can apply the Sobolev Embedding Theorem, see for example [1], to deduce $K\zeta \in C^1(\overline{\Omega}_{n_0})$. Also, since $\gamma(K\zeta, \partial\Omega, W^{1,p}(\Omega_{n_1})) = 0$, we infer existence of a sequence $\{\omega_n\}_{n=1}^{\infty} \subset C^1(\Omega_{n_1})$ such that ω_n vanish near $\partial\Omega$ and $\omega_n \to K\zeta$ in $W^{1,p}(\Omega_{n_1})$, as $n \to \infty$. Now, by the Trace Theorem, we have

$$trace(\omega_n) \to trace(K\zeta), \text{ in } L^2(\partial\Omega_{n_1}, \mathcal{H}^1(\partial\Omega_{n_1})),$$

as $n \to \infty$, where $\mathcal{H}^1(\partial\Omega_{n_1})$ denotes the Hausdorff measure of dimension one restricted to $\partial\Omega_{n_1}$. In particular, we obtain

$$trace(\omega_n) \to trace(K\zeta), \text{ in } L^2(\partial\Omega, \mathcal{H}^1(\partial\Omega)),$$

as $n \to \infty$. Since ω_n vanish near $\partial\Omega$ we deduce that trace$(K\zeta)$ vanishes on $\partial\Omega$, almost everywhere with respect to $\mathcal{H}^1(\partial\Omega)$; but $K\zeta \in C^1(\overline{\Omega}_{n_0})$, hence $K\zeta$ vanishes on $\partial\Omega$, pointwise. So we are done. $\diamondsuit$

Remark. Note that from (4.9) and Agmon's regularity theory, see [2], it follows that $K\zeta \in W^{2,p}_{loc}(\Omega)$. Therefore, by the Sobolev Embedding Theorem, we infer $K\zeta \in C^1(\Omega)$; since $K\zeta \in C^1(\overline{\Omega}_{n_0})$ we infer $K\zeta \in C^1(\overline{\Omega})$. Therefore $K\zeta$ represents the stream function of a flow tangential to $\partial\Omega$, see the remark following Lemma 4 (below).

4.3.3 The existence of η_c

In this section we investigate the existence of $\eta_c := \eta_1 + x_2^2 + c\log|x|$, $c \in \mathbb{R}$, such that $\eta_c \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and

$$\begin{cases} \Delta\eta_c &= & 2, \text{ in } \Omega \\ \eta_c &= & 0, \text{ on } \partial\Omega \\ \nabla\eta_c &\to & (0, 2x_2), \text{ as } |x| \to \infty. \end{cases}$$

To find η_c, it clearly suffices to prove the following

Lemma 4 *There exists $\eta_1 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that*

$$\begin{cases} \Delta \eta_1 &= 0 \ \mathit{in}\ \Omega \\ \eta_1 &= -x_2^2 - c\log|x| \ \mathit{on}\ \partial\Omega \\ \nabla \eta_1 &\to 0 \ \mathit{as}\ |x| \to \infty. \end{cases}$$

Proof. Let $\alpha := \mathrm{Min}\{-x_2^2 - c\log|x| \mid x \in \partial\Omega\}$ and set $k := -\alpha^-$, where α^- denotes the negative part of α. Next we denote by u_n the classical solution of the following Dirichlet problem

$$\begin{cases} \Delta u_n &= 0, \ \text{in}\ \Omega_n \\ u_n &= -x_2^2 - c\log|x|, \ \text{on}\ \partial\Omega_n \\ u_n &= k, \ \text{on}\ \partial B_n. \end{cases}$$

By an application of the Maximum Principle we obtain

$$k \le u_n \le \mathrm{Max}\{-x_2^2 - c\log|x| \mid x \in \partial\Omega\} := \beta. \tag{4.11}$$

Again by applying the Maximum Principle we infer $u_{n+1} \ge u_n$, in $\overline{\Omega}_n$, for every $n \in \mathbb{N}$. Indeed, we have $\Delta(u_{n+1} - u_n) = 0$, in Ω_n, and $u_{n+1} - u_n = 0$, on $\partial\Omega$ and finally, $u_{n+1} - u_n \ge 0$, on ∂B_n, from (4.11). Therefore, it makes sense to define:

$$\eta_1(x) := \lim_{n\to\infty} u_n(x). \tag{4.12}$$

From [22, p151] we infer that η_1 is harmonic in Ω and u_n converges uniformly on compact subsets of Ω. This, in turn, implies that $\eta_1 \in C^2(\Omega) \cap C(\overline{\Omega})$, see [22, Theorem 6, p149]. We now show $\nabla\eta_1 \to 0$, as $|x| \to \infty$. To do this, we apply Harnack's inequality [28]:

$$\begin{aligned} |\nabla \eta_1(x)| &\le \left(\frac{4}{|x|-1}\right)^2 \|\eta_1\|_{\infty,\Omega} \\ &\le \frac{16\beta}{(|x|-1)^2}, \ \forall x \in \Omega,\ |x| > 1. \end{aligned}$$

Hence, $\nabla\eta_1 \to 0$, as $|x| \to \infty$. It then remains to show that, in fact, $\eta \in C^1(\overline{\Omega})$. Note that it suffices to prove $\eta \in C^1(\overline{\Omega}_n)$, for any $n \in \mathbb{N}$. This readily follows from [17, Proposition 2, p461], since $\partial\Omega_n$ is a regular manifold of class $W^{2,\infty}$, see[17, p461].$\diamondsuit$

Remark. Physically η_c represents the stream function of a flow with constant vorticity which is tangential to $\partial\Omega$ and approaches a *shear* state at infinity. Indeed we have

$\eta_c \in C^2(\Omega) \cap C(\overline{\Omega})$ and

$$\begin{cases} \Delta\eta_c &= & 2 \ \text{ in } \Omega \\ \eta_c &= & 0 \ \text{ on } \partial\Omega \\ \nabla\eta_c &\to & (0, 2x_2) \ \text{ as } |x| \to \infty. \end{cases}$$

The fact that η_c vanishes on $\partial\Omega$ implies that the flow is tangential to $\partial\Omega$. This follows from the following elementary calculations. Let us assume that $\partial\Omega$ is given by the following parametrisation:

$$x_1 = x_1(s) \quad x_2 = x(s), \ \ s \in (a, b).$$

Since $\eta_c = 0$ on $\partial\Omega$, we obtain

$$\eta_c(x_1(s), x_2(s)) = 0, \ \ s \in (a, b).$$

This, in turn, implies that

$$\frac{d}{ds}\eta_c(x_1(s), x_2(s)) = 0, \ \ s \in (a, b),$$

since $\eta_c \in C^1(\overline{\Omega})$. Hence

$$\nabla\eta_c(x_1(s), x_2(s)) \cdot (x_1'(s), x_2'(s)) = 0, \ \ s \in (a, b).$$

Hence $\nabla\eta_c(x_1(s), x_2(s))$ and $\vec{n}(x_1(s), x_2(s))$, the exterior unit normal at $(x_1(s), x_2(s))$, are linearly dependent. Therefore $\nabla^{\perp}\eta_c(x_1(s), x_2(s))$ is perpendicular to $\vec{n}(x_1(s), x_2(s))$ for all s in (a, b).

4.3.4 A result from Burton's Theory

In this section we state a simple version of [11], namely,

Lemma 5 *Let $g \in L^{p^*}(\Omega_n) \cap W^{2,1}_{loc}(\Omega_n)$. Suppose that $\hat{\zeta}$ maximises $\int_{\Omega_n} \zeta g$, relative to $\zeta \in \overline{\mathcal{F}(\Omega_n)^w}$ and that $-\Delta g \geq \hat{\zeta}$ almost everywhere in Ω_n. Then $\hat{\zeta} \in \mathcal{F}(\Omega_n)$ and there is an increasing function ϕ such that $\hat{\zeta} = \phi \circ g$, almost everywhere in Ω_n.*

4.4 The main result

In this section we present the proof of Theorem 1 followed by the physical interpretation of our result.

Proof of Theorem 1. As noted in section (3.2.2) we first need to consider (P_n). Let us fix an integer $n > (1+a^2)^{1/2}$. From Lemma 3 it follows that $\Psi(\zeta)$ is weakly sequentially continuous on the linear space of functions in $L^p(\Omega)$ which vanish outside Ω_n. On the other hand since $\overline{\mathcal{F}(\Omega_n)^w}$ is weakly sequentially compact we infer that $\Psi(\zeta)$ attains its maximum relative to $\zeta \in \overline{\mathcal{F}(\Omega_n)^w}$. Let ζ_n denote a corresponding maximiser. To show that, in fact, $\zeta_n \in \mathcal{F}(\Omega_n)$ we apply a standard argument as follows. Consider $\zeta \in \overline{\mathcal{F}(\Omega_n)^w}$ and $t \in (0,1]$. Then $(1-t)\zeta_n + t\zeta \in \overline{\mathcal{F}(\Omega_n)^w}$, by convexity of $\overline{\mathcal{F}(\Omega_n)^w}$. So we have

$$\Psi(\zeta_n) \geq \Psi((1-t)\zeta_n + t\zeta) = \Psi(\zeta_n) + tD\Psi[\zeta_n](\zeta - \zeta_n) + o(t),$$

as t tends to zero, whence

$$D\Psi[\zeta_n](\zeta - \zeta_n) \leq 0. \tag{4.13}$$

From (4.13) and the fact that $D\Psi[\zeta_n]$ can be identified with $K\zeta_n - \eta_c$ we deduce that ζ_n maximises the linear functional

$$\int_\Omega \zeta K\zeta_n,$$

relative to $\zeta \in \overline{\mathcal{F}(\Omega_n)^w}$. Also from Lemma 3 we have that $-\Delta K\zeta_n = \zeta_n$ almost everywhere in Ω_n. Thus $-\Delta(K\zeta_n - \eta_c) = \zeta_n + 2$ almost everywhere in Ω_n. Now we are in a position to apply Lemma 5 to derive $\zeta_n \in \mathcal{F}(\Omega_n)$; moreover, the existence of an increasing function ϕ such that

$$\zeta_n = \phi \circ (K\zeta_n - \eta_c), \text{ a.e. in } \Omega_n, \tag{4.14}$$

is ensured. From (4.14) it follows that there exists $\gamma_n \in \mathbb{R}$ such that

$$\operatorname{supp}(\zeta_n) = \{x \in \Omega_n \mid K\zeta_n(x) - \eta_c(x) \geq \gamma_n\}, \text{ a.e. in } \Omega_n. \tag{4.15}$$

In the next step we show that γ_n can not be too negative. For this purpose we recall that η_1 is bounded on $\overline{\Omega}$, see (4.11). Without loss of generality we may assume that $\gamma_n \leq -\|\eta_1\|_\infty$ since otherwise we are done. Let B denote the ball centered at the origin with radius $(|\gamma_n| - \|\eta_1\|_\infty)^{1/2}/(1+c)^{1/2}$. For $x \in B$ we have

$$\begin{aligned} K\zeta_n(x) - \eta_c(x) &\geq -\|\eta_1\|_\infty - x_2^2 - c\log|x| \\ &\geq -(\|\eta_1\|_\infty + (1+c)|x|^2) \\ &\geq -|\gamma_n| = \gamma_n \end{aligned}$$

Therefore, $B \subseteq \operatorname{supp}(\zeta_n)$, modulo a set of measure zero. Whence $|B| \leq |\operatorname{supp}(\zeta_n)|$. This, in turn, implies

$$\gamma_n \geq -(\|\eta_1\|_\infty + (1+c)a^2) := \hat{\gamma}.$$

Therefore, we deduce

$$\operatorname{supp}(\zeta_n) \subseteq \{x \in \Omega_n \mid K\zeta_n(x) - \eta_c(x) \geq \hat{\gamma}\},$$

modulo a set of measure zero. For $x \in \{x \in \Omega_n \mid K\zeta_n(x) - \eta_c(x) \geq \hat{\gamma}\}$ we have

$$\begin{aligned}
\hat{\gamma} \leq K\zeta_n(x) - \eta_c(x) &\leq K_R\zeta_n(x) - \eta_c(x) \\
&= K_R\zeta_n(x) - \eta_1(x) - x_2^2 - c\log|x| \\
&\leq K_R\zeta_n(x) - \frac{c-1}{2\pi}\log|x| - k - \frac{(2\pi-1)c+1}{2\pi}\log|x|.
\end{aligned}$$

Since $\|\zeta_n\|_1 = 1$ we can write

$$K_R\zeta_n(x) - \frac{c-1}{2\pi}\log|x| = \frac{1}{2\pi}\int_{\mathbb{R}^2\setminus\overline{B_R}} \log\frac{R^{-1}|x||y-x^*|}{|x|^{c-1}|y-x|}\zeta_n(y)dy. \tag{4.16}$$

We now proceed to estimate the integral in (4.16). From (†) we have

$$\log\frac{R^{-1}|x||y-x^*|}{|x|^{c-1}|y-x|} \leq \begin{cases} \log 4R^{-1}/|x|^{c-2}, & |y| \geq 2|x|, \\ \log(2R^{-1}|x|^2+1)/|x|^{c-1}|y-x|, & |y| \leq 2|x|. \end{cases}$$

Since $c \geq 3$ and $|x| > R$ we obtain

$$\log\frac{R^{-1}|x||y-x^*|}{|x|^{c-1}|y-x|} \leq \begin{cases} \log 4R^{1-c}, & |y| \geq 2|x|, \\ \log(2R^{2-c}+R^{-1})/|y-x|, & |y| \leq 2|x|. \end{cases}$$

Therefore, by [31], we obtain

$$\frac{1}{2\pi}\int_{\mathbb{R}^2\setminus\overline{B_R}} \log\frac{R^{-1}|x||y-x^*|}{|x|^{c-1}|y-x|}\zeta_n(y)dy \leq C,$$

where C is a universal constant. Whence

$$\hat{\gamma} \leq C - k - \frac{(2\pi-1)c+1}{2\pi}\log|x|.$$

This, in turn, yields

$$|x| \leq e^{(C-k-\hat{\gamma})/\hat{c}} := \hat{r}, \tag{4.17}$$

where $\hat{c} := ((2\pi - 1)c + 1)/2\pi$. From (4.17) it follows that

$$\operatorname{supp}(\zeta_n) \subseteq B_{\hat{r}}, \ \ \forall n \in \mathbb{N},$$

where $n > (1 + a^2)^{1/2}$, modulo sets of measure zero. Therefore, if $r^* := [\hat{r}] + 1$, then any $\zeta_{r^*} \in \Sigma_{r^*}$ is indeed a solution to (P).

Now let $\zeta \in \Sigma$. It is clear that

$$K\zeta(x) - \eta_c(x) \to -\infty, \ \ \text{as } |x| \to \infty.$$

Hence, there exists $n^* \in \mathbb{N}$ such that $\operatorname{supp}(\zeta) \subseteq \Omega_{n^*}$, except on a set of measure zero, and

$$K\zeta(x) - \eta_c(x) < \hat{\gamma}, \ \ |x| > n^*.$$

Since $\zeta \in \Sigma$ we clearly have $\zeta \in \Sigma_{n^*}$. Therefore by Lemma 5, there exists an increasing function $\hat{\phi}$ such that

$$\zeta = \hat{\phi} \circ (K\zeta - \eta_c), \ \ \text{a.e. in } \Omega_{n^*}.$$

Now we define

$$\phi(t) := \begin{cases} \hat{\phi}(t), & t \geq \hat{\gamma}, \\ 0, & t < \hat{\gamma}. \end{cases}$$

Therefore, we obtain

$$\zeta = \phi \circ (K\zeta - \eta_c), \ \ \text{a.e. in } \Omega.$$

Finally, since $-\Delta K\zeta = \zeta$ almost everywhere in Ω, by Lemma 3, we deduce that

$$-\Delta K\zeta = \phi \circ (K\zeta - \eta_c), \ \ \text{a.e. in } \Omega,$$

and we are done.◇

Remark. The differential equation in (iii) is indeed the Euler-Lagrange equation for (P); proof of this fact is sketched as follows: If $\hat{\zeta} \in \Sigma$ then by the weak semicontinuity of Ψ we have

$$\Psi(\hat{\zeta}) = \sup_{\zeta \in \overline{\mathcal{F}(\Omega_n)^w}} \Psi(\zeta),$$

for a sufficiently large n. Hence, by setting $\hat{\Psi} = -\Psi$, we obtain

$$\hat{\Psi}(\hat{\zeta}) = \inf_{\zeta \in \overline{\mathcal{F}(\Omega_n)^w}} \hat{\Psi}(\zeta).$$

Now by [16, p52] we deduce

$$0 \in \partial^* \hat{\Psi}[\hat{\zeta}] + N_{\overline{\mathcal{F}(\Omega_n)^w}}(\hat{\zeta}),$$

where ∂^* and $N_{\overline{\mathcal{F}(\Omega_n)^w}}(\hat{\zeta})$ denote the generalised subgradient operator and the normal cone to $\overline{\mathcal{F}(\Omega_n)^w}$ at $\hat{\zeta}$, respectively, see [16] for appropriate definitions. From [16, Proposition 2.4.4] and the convexity of $\overline{\mathcal{F}(\Omega_n)^w}$ we infer $N_{\overline{\mathcal{F}(\Omega_n)^w}}(\hat{\zeta}) = \partial\delta_{\overline{\mathcal{F}(\Omega_n)^w}}(\hat{\zeta})$, where

$$\delta_{\overline{\mathcal{F}(\Omega_n)^w}}(\zeta) := \begin{cases} 0 & \zeta \in \overline{\mathcal{F}(\Omega_n)^w} \\ +\infty & \text{otherwise,} \end{cases}$$

and $\partial\delta_{\overline{\mathcal{F}(\Omega_n)^w}}(\hat{\zeta})$ stands for the subdifferential of $\delta_{\overline{\mathcal{F}(\Omega_n)^w}}$ at $\hat{\zeta}$, that is,

$$\partial\delta_{\overline{\mathcal{F}(\Omega_n)^w}}(\hat{\zeta}) := \{\xi \in L^{p^*}(\Omega_n) |\ \delta_{\overline{\mathcal{F}(\Omega_n)^w}}(\zeta) \geq \delta_{\overline{\mathcal{F}(\Omega_n)^w}}(\hat{\zeta}) + \xi(\zeta - \hat{\zeta}),\ \forall \zeta \in L^p(\Omega_n)\}.$$

Observe that $\Psi \in C^1(L^p(\Omega_n))$, hence by a standard result, see for example [16], it follows that $\partial^*\hat{\Psi}(\hat{\zeta}) = \{-D\Psi[\hat{\zeta}]\}$, the Frechet- derivative of Ψ at $\hat{\zeta}$. Therefore, there exists ξ in $\partial\delta_{\overline{\mathcal{F}(\Omega_n)^w}}(\hat{\zeta})$ such that

$$D\Psi[\hat{\zeta}](\zeta - \hat{\zeta}) = \xi(\zeta - \hat{\zeta}),\ \ \forall \zeta \in \overline{\mathcal{F}(\Omega_n)^w}.$$

This, in turn, implies that $D\Psi[\hat{\zeta}](\zeta - \hat{\zeta}) \leq 0$. Simple calculations show that $D\Psi[\hat{\zeta}]$ can be identified with $K\hat{\zeta} - \eta_c$. Whence $\hat{\zeta}$ maximises the linear functional $\int_{\Omega_n} \zeta\ D\Psi[\hat{\zeta}]$ relative to $\zeta \in \overline{\mathcal{F}(\Omega_n)^w}$. Now by Lemma 3 we have $-\Delta D\Psi[\hat{\zeta}] = \hat{\zeta}+2$ almost everywhere in Ω_n. On the other hand an application of Lemma 5 yields

$$\hat{\zeta} = \phi^\dagger \circ (\psi - \eta_c),\ \text{a.e. in } \Omega_n,$$

for some increasing function $\phi^\dagger$. Finally, by an extension of $\phi^\dagger$, see the proof of Theorem 1, we construct a function ϕ such that

$$\hat{\zeta} = \phi \circ (\psi - \eta_c),\ \text{a.e. in } \Omega.$$

Hence (iii) follows.◇

Let us note that the function $\psi = K\zeta$ constructed in Theorem 1 gives rise to a flow which is tangential to $\partial\Omega$ and approaches a shear state at infinity. Indeed if we denote the stream function of the flow by $\psi - \eta_c$, it is clear that the flow is tangential since $\psi - \eta_c$ vanishes on $\partial\Omega$. To show the shear-behaviour at long range points, it suffices to prove

$$\nabla K\zeta \to 0 \ \text{ as } |x| \to \infty. \tag{4.18}$$

To prove (4.18) we proceed as follows. Suppose $\text{supp}(\zeta) \subset B_l$. The function $u(x) := K_R\zeta(x) - K\zeta(x)$ is clearly harmonic and non-negative in $\mathbb{R}^2 \setminus \overline{B_1}$. Thus, by [28, p29], we obtain

$$|\nabla u(x)| \leq \frac{2}{|x|-1} u(x), \ \ x \in \mathbb{R}^2 \setminus \overline{B_1}.$$

Hence

$$|\nabla K\zeta(x)| \leq \frac{2}{|x|-1} u(x) + |\nabla K_R\zeta(x)|, \ \ x \in \mathbb{R}^2 \setminus \overline{B_1}.$$

Since $K\zeta$ is non-negative we infer that $u(x) \leq K_R\zeta(x)$ for all $x \in \mathbb{R}^2 \setminus \overline{B_1}$. On the other hand, elementary calculations yield

$$K_R\zeta(x) \leq \frac{1}{2\pi} \log R^{-1}|x|, \ \ |x| > 2l.$$

Hence

$$|\nabla K\zeta(x)| \leq \frac{1}{\pi(|x|-1)} \log R^{-1}|x| + |\nabla K_R\zeta(x)|, \ \ |x| > \text{Max}\{2l, 1\}.$$

Therefore, to derive (4.18) it suffices to show

$$\nabla K_R\zeta(x) \to 0, \ \text{ as } |x| \to \infty. \tag{4.19}$$

For this purpose we apply Harnack's inequality [28, p27] as follows. Consider $x \in \mathbb{R}^2 \setminus \overline{B}_{2l}$, then

$$|\nabla K_R\zeta(x)| \leq \frac{4}{|x|-l} \sup_{z \in \overline{B}_l(x)} |K_R\zeta(z)|. \tag{4.20}$$

Since $||\zeta||_1 = 1$ we readily derive

$$K_R\zeta(z) \leq \frac{1}{2\pi} \log R^{-1}l|z| + \frac{1}{2\pi} \int_{\mathbb{R}^2 \setminus \overline{B}_R} \log \frac{1}{|y-z|} \zeta(y)dy, \ \ z \in \overline{B}_l(x).$$

By an application of [31] we find

$$\frac{1}{2\pi} \int_{\mathbb{R}^2 \setminus \overline{B}_R} \log \frac{1}{|y-z|} \zeta(y)dy \leq C,$$

where C is some universal constant. Therefore, since $|z| \leq |x| + l$ for $z \in \overline{B}_l(x)$, we obtain

$$K_R\zeta(z) \leq \frac{1}{2\pi} \log R^{-1}l(|x|+l) + C, \ \ z \in \overline{B}_l(x). \tag{4.21}$$

It is now clear that (4.19) follows from (4.20) and (4.21).

Finally, the circulation of the flow around $\partial\Omega$, that is, $Cir(K\zeta - \eta_c, \partial\Omega)$, is a function of $c \geq 3$, denoted by $\xi(c)$. Hence we have proved the following

Corollary There exists a flow in Ω containing a bounded vortex anomaly, tangential to $\partial\Omega$ with a shear-behaviour at long range points; moreover, the circulation can be prescribed in terms of $\xi(c)$, provided $c \geq 3$.

Chapter 5

Existence of a steady flow with a bounded vortex in an unbounded domain (part one)

5.1 Introduction

In this chapter we prove the existence of steady 2-dimensional ideal fluid flows occupying Π_+ (the first quadrant) and containing a bounded vortex. Such a flow will be described by a stream function $\psi : \Pi_+ \to \mathbb{R}$. At infinity we will have $\psi \to -\lambda x_1 x_2$ which is the stream function for an irrotational flow with velocity field $-\lambda(x_1, -x_2)$. The vorticity is given by $-\Delta\psi$, where Δ is the Laplacian, and $-\Delta\psi$ vanishes outside a bounded region avoiding the boundary of Π_+. It will be proved that $-\Delta\psi = \phi \circ \psi$ is satisfied almost everywhere in Π_+ for ϕ an increasing function, unknown a priori. In our result the vorticity field $\zeta(=-\Delta\psi)$ is a rearrangement of a prescribed non-negative, non-trivial function ζ_0 having bounded support. The existence theorem is proved by maximising a functional over the set of rearrangements of ζ_0, vanishing outside bounded sets in Π_+, in order to obtain the vorticity field ζ. This variational principle was adapted by Burton [12] from one for vortex rings in 3 dimensions, proposed by Benjamin[6].

Lack of compactness caused by the unboundedness of the domain of interest is the motivation to use the strategy proposed by Benjamin [6], as in the preceding chapters.

This chapter and the next one have been inspired by [12]; the main difference is that in our case the domain is not symmetric, hence we can not make use of Steiner-symmetrisation, as in [12], in our analysis, prescribing an "impulse functional" rather than λ.

This very same problem will be taken up in the following chapter where we use a different variational principle.

5.2 Notation, definitions and statement of results

Henceforth p denotes a real number in $(2,\infty)$ and $p^* := p/(p-1)$. The upper and the right half planes are designated by Π_u and Π_r, respectively; and the first quadrant by Π_+. Generic points of $\mathbb{R}^2$ are denoted by $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2)$, etc. For $x \in \mathbb{R}^2$ we let $\overline{x}$, $\underline{x}$, $\underline{\overline{x}}$ to denote the reflections of x about the x_1-axis, x_2-axis and the origin, respectively. For $\xi > 0$ we define $\Pi_+(\xi) := \{x \in \mathbb{R}^2 \mid 0 < x_1 < \xi,\ 0 < x_2 < \xi\}$. In this chapter we will be dealing with three different Green's functions, namely, the Green's functions for $-\Delta$ with homogeneous Dirichlet boundary conditions on Π_u, Π_r and Π_+; it is a standard result that these functions are defined as follows, respectively,

$$
\begin{aligned}
G_u(x,y) &:= \frac{1}{2\pi}\log\frac{|x-\overline{y}|}{|x-y|}, && x, y \in \Pi_u,\ x \neq y,\\
G_r(x,y) &:= \frac{1}{2\pi}\log\frac{|x-\underline{y}|}{|x-y|}, && x, y \in \Pi_r,\ x \neq y,\\
G_+(x,y) &:= \frac{1}{2\pi}\log\frac{|x-\overline{y}||x-\underline{y}|}{|x-y||x-\underline{\overline{y}}|}, && x, y \in \Pi_+,\ x \neq y.
\end{aligned}
$$

Accordingly, for measurable functions ζ on $\mathbb{R}^2$, we define the following integral operators

$$
\begin{aligned}
K_u\zeta(x) &:= \int_{\Pi_u} G_u(x,y)\zeta(y)dy,\\
K_r\zeta(x) &:= \int_{\Pi_r} G_r(x,y)\zeta(y)dy,\\
K_+\zeta(x) &:= \int_{\Pi_+} G_+(x,y)\zeta(y)dy,
\end{aligned}
$$

whenever the integrals exist.

Remark. In the following section it will be shown that if $\zeta \in L^p(\mathbb{R}^2)$ vanishes outside a bounded set then these integrals are defined everywhere in $\mathbb{R}^2$.

For a measurable set A in $\mathbb{R}^2$, we use $|A|$ to denote the 2-dimensional Lebesgue measure of A. The strong support of a measurable function ζ, denoted $\mathrm{supp}(\zeta)$, is defined as follows

$$\mathrm{supp}(\zeta) := \{x \in \mathrm{dom}(\zeta) \mid \zeta(x) > 0\}.$$

To define the rearrangement class needed for our variational problem we fix a non-negative, non-trivial function $\zeta_0 \in L^p(\mathbb{R}^2))$ which vanishes outside a bounded set; in addition we assume

$$|\text{supp}(\zeta_0)| = \pi a^2, \tag{5.1}$$

for some $a > 0$. Now we denote the set of rearrangements of ζ_0, on $\mathbb{R}^2$, which vanish outside bounded subsets of Π_+ by $\mathcal{F}$. The subset of $\mathcal{F}$ comprising functions vanishing outside $\Pi_+(\xi)$ are designated by $\mathcal{F}(\xi)$, where it is assumed that $\xi \geq a\pi^{1/2}$ to ensure $\mathcal{F}(\xi) \neq \emptyset$.

Next we define the *energy*; for $v \in L^p(\Pi_+)$ having bounded support and $\lambda \in \mathbb{R}$, we define

$$\begin{aligned}\Psi(v) &:= \frac{1}{2}\int_{\Pi_+} vKv,\\ \Im(v) &:= \int_{\Pi_+} x_1x_2v\end{aligned}$$

and the kinetic energy

$$\Psi_\lambda(v) := \Psi(v) - \lambda\Im(v),$$

whenever the integrals exist. Now we are in a position to define the variational problem (P_λ) as follows

$$(P_\lambda): \quad \sup_{\zeta\in\mathcal{F}} \Psi_\lambda(\zeta).$$

The set of solutions of (P_λ) is denoted by Σ_λ. For $\xi > a\pi^{1/2}$ the truncated variational problem $(P_\lambda(\xi))$ is defined by

$$(P_\lambda(\xi)): \quad \sup_{\zeta\in\mathcal{F}(\xi)} \Psi_\lambda(\zeta),$$

and accordingly, $\Sigma_\lambda(\xi)$ denotes the set of solutions.

In this chapter we prove

Theorem 1 *There exists $\lambda_0 > 0$ such that $\Sigma_\lambda \neq \emptyset$, for $\lambda \in (0, \lambda_0)$. Moreover, if $\zeta \in \Sigma_\lambda$ and $\psi := K_+\zeta$, then*

$$-\Delta\psi = \phi \circ (\psi - \lambda x_1x_2), \quad \textit{a.e. in } \Pi_+, \tag{5.2}$$

for some increasing function ϕ.

5.3 Preliminary results

In this section we state and prove some of the properties of the operator K_+ and consequently show that the functional Ψ_λ considered on $L^p(\Pi_+(\xi))$, the subset of $L^p(\Pi_+)$

comprising functions that vanish outside $\Pi_+(\xi)$, is strictly convex and weakly sequentially continuous. This, in turn, lays out the foundations for the application of Burton's theory to prove solvability of $(P_\lambda(\Pi_+(\xi)))$.

5.3.1 The operator K_+ and some estimates

Lemma 1 *Let $\zeta \in L^p(\Pi_+)$ vanishes outside a bounded set. Then*
(i) $K_+\zeta \in C^1(\mathbb{R}^2)$.
(ii) $|\nabla K_+\zeta(x)| \leq C\|\zeta\|_p$, for every $x \in \mathbb{R}^2$, where C depends on $|supp(\zeta)|$ and p.
(iii) $|K_+\zeta(x)| \leq C min\{|x_1|, |x_2|\}\|\zeta\|_p$, for every $x \in \Pi_+$, where C is the constant in (ii).

Proof. (i) is an immediate consequence of a result about Newtonian potentials of densities with compact support. Specifically, let

$$N\zeta_e(x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{1}{|x-y|}\zeta_e(y)dy$$

denote the Newtonian potential of the zero-extension, to all of $\mathbb{R}^2$, of ζ. Since $p > 2$ and ζ has compact support we can apply [24, Lemmas A.7 and A.9] to deduce $N\zeta_e \in C^1(\mathbb{R}^2)$ and

$$\nabla N\zeta_e(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \nabla_x \log\frac{1}{|x-y|}\,\zeta_e(y)dy,\ \ \forall x \in \mathbb{R}^2. \tag{5.3}$$

Clearly we have

$$K_+\zeta(x) = N\zeta_e(x) + N\zeta_e(\underline{\overline{x}}) - N\zeta_e(\overline{x}) - N\zeta_e(\underline{x}).$$

Hence, $K_+\zeta \in C^1(\mathbb{R}^2)$; moreover it is obvious that the truth of (ii) will emerge once we show

$$|\nabla N\zeta_e(x)| \leq C\ \|\zeta\|_p,\ \ \forall x \in \mathbb{R}^2, \tag{5.4}$$

where C is a constant depending on $|\mathrm{supp}(\zeta)|$ and p. To do this, we use (5.3) to deduce

$$|\nabla N\zeta_e(x)| \leq \int_{\mathbb{R}^2} \frac{1}{|x-y|}\ |\zeta(y)|dy,\ \forall x \in \mathbb{R}^2.$$

Now let us fix $x \in \mathbb{R}^2$ and denote the Schwarz-rearrangement of $|\zeta|$, about x, by ζ^*. Therefore, by a standard inequality, see for example [31], we obtain

$$|\nabla N\zeta_e(x)| \leq \frac{1}{2\pi}\int_{B_l(x)} \frac{1}{|x-y|}\ \zeta^*(y)dy,$$

where $l := (|\mathrm{supp}(\zeta)|/\pi)^{1/2}$. Hence, by an application of Hölder's inequality we derive

$$|\nabla N\zeta_e(x)| \leq \frac{1}{2\pi}\left(\int_{B_l(x)} \frac{1}{|x-y|^{p^*}}dy\right)^{1/p^*} ||\zeta||_p. \tag{5.5}$$

Elementary calculations accompanied by application of polar co-ordinates yield

$$\int_{B_l(x)} \frac{1}{|x-y|^{p^*}}dy \leq C,$$

where C depends only on l and p. So we derive (5.4), hence (ii). Now to derive (iii) we fix $x \in \Pi_+$. Since $K_+\zeta \in C^1(\mathbb{R}^2)$ and vanishes on the boundary of Π_+, we can apply the Mean Value Theorem to obtain

$$|K_+\zeta(x)| = |K_+\zeta(x) - K_+\zeta(x_1, 0)| \leq x_2|\nabla K_+\zeta(\hat{x})|,$$

where $\hat{x}$ is a point on the segment joining x to $(x_1, 0)$. Whence from (ii) we deduce $|K_+\zeta(x)| \leq C\ x_2\ ||\zeta||_p$. Similarly, one can show $|K_+\zeta(x)| \leq C\ x_1\ ||\zeta||_p$, so we derive (iii) as required. $\diamondsuit$

Lemma 2 *Let U be an open and bounded subset of Π_+. Then for every $q \geq 1$, $K_+ : L^p(U) \to L^q(U)$ is a compact linear operator.*

Proof. The well-definedness of $K_+ : L^p(U) \to L^q(U)$ follows from Lemma 1(iii). The linearity of K_+ follows from the definition. To show compactness of K_+ it suffices to show that $K_+ : L^p(U) \to W^{1,2}(U)$ is bounded, since then by an application of the Sobolev Embedding Theorem we derive the desired result. Let us now fix $\zeta \in L^p(\Pi_+)$ that vanishes outside U. Then by applying Lemma 1(ii),(iii) we infer $||K_+\zeta||_2 \leq C||\zeta||_p$ and $||\nabla K_+\zeta||_2 \leq C||\zeta||_p$, hence

$$||K_+\zeta||_{W^{1,2}(U)} \leq C\ ||\zeta||_p.$$

Therefore we are done. $\diamondsuit$

Lemma 3 *Let $\zeta \in L^p(\Pi_+)$ vanishes outside a bounded set. Then*

$$\nabla K_+\zeta(x) = O(|x|^{-2}), \qquad K_+\zeta(x) = O(|x|^{-1}),\ |x| \to \infty.$$

Proof. This is an immediate consequence of [12, Lemma 7] where it is proved that

$$\nabla K_u\zeta(x) = O(|x|^{-2}), \qquad K_u\zeta(x) = O(|x|^{-1}),\ |x| \to \infty.$$

Therefore, the conclusion of the lemma follows from the identity

$$K_+\zeta(x) = K_u\zeta(x) - K_u\zeta(\underline{x}), \quad x \in \Pi_+.$$

Lemma 4 *Let q and U be as in Lemma 2. Then $K_+ : L^p(U) \to L^q(U)$ is strictly positive, that is, for every non-trivial function $\zeta \in L^p(\Pi_+)$ vanishing outside U,*

$$\int_{\Pi_+} \zeta K_+\zeta > 0.$$

Proof. Let us fix $\zeta \in L^p(\Pi_+)$ vanishing outside U. Then, from [12, Lemma 3(i)], we have

$$-\Delta K_u\zeta = \zeta, \quad \text{in } \mathcal{D}'(\Pi_u).$$

Hence, we also have $-\Delta K_u\zeta = \zeta$, in $\mathcal{D}'(\Pi_+)$, since $K_+\zeta(x) = K_u\zeta(x) - K_u\zeta(\underline{x})$ for all $x \in \mathbb{R}^2$. Now by Agmon's regularity theory [2] we deduce $K_+\zeta \in W^{2,p}_{loc}(\mathbb{R}^2)$. In particular, $K_+\zeta \in W^{2,p}_{loc}(\overline{\Pi}_+)$. Therefore, in fact, we have $-\Delta K_u\zeta = \zeta$ almost everywhere in Π_+. Next we let $\Omega(R) := B_R \cap \Pi_+$; hence the boundary of $\Omega(R)$ being Lipschitz we can apply the weak Divergence Theorem, see for example [29], to obtain

$$-\int_{\Omega(R)} \zeta K_+\zeta + \int_{\Omega(R)} |\nabla K_+\zeta|^2 = \int_{\partial\Omega(R)} \gamma(K_+\zeta)\, \gamma(\partial_{\vec{n}}K_+\zeta) d\sigma,$$

where γ stands for the trace operator on $\partial\Omega(R)$ and $\vec{n}$ denotes the unit outward normal vector to $\partial\Omega(R)$. Since $K_+\zeta \in C^1(\overline{\Pi}_+)$ we have

$$\int_{\partial\Omega(R)} \gamma(K_+\zeta)\, \gamma(\partial_{\vec{n}}K_+\zeta) d\sigma = \int_{\partial\Omega(R)} (K_+\zeta)\ (\partial_{\vec{n}}K_+\zeta)\ d\sigma.$$

Therefore

$$-\int_{\Omega(R)} \zeta K_+\zeta + \int_{\Omega(R)} |\nabla K_+\zeta|^2 = \int_{\partial\Omega(R)} (K_+\zeta)\ (\partial_{\vec{n}}K_+\zeta)\ d\sigma.$$

Now from Lemma 3 we infer

$$\lim_{R\to\infty} \int_{\partial\Omega(R)} (K_+\zeta)\ (\partial_{\vec{n}}K_+\zeta)\ d\sigma = 0.$$

Moreover, since $\int_{\Pi_+} \zeta K_+\zeta$ is finite and $|\nabla K_+\zeta|$ is bounded in $\mathbb{R}^2$ we may apply the Lebesgue Dominated Convergence Theorem to conclude

$$\lim_{R\to\infty} \int_{\Omega(R)} \zeta\ K_+\zeta = \int_{\Pi_+} \zeta\ K_+\zeta, \quad \lim_{R\to\infty} \int_{\Omega(R)} |\nabla K_+\zeta|^2 = \int_{\Pi_+} |\nabla K_+\zeta|^2.$$

Therefore, we derive $\int_{\Pi_+} \zeta\, K_+\zeta = \int_{\Pi_+} |\nabla K_+\zeta|^2$ and we are done.$\diamondsuit$

5.3.2 Burton's theory and problem $P_\lambda(\xi)$

The following lemma is a result from Burton's theory [10]

Lemma 5 *Suppose $\Phi : L^p(\Pi_+(\xi)) \to \mathbb{R}$ is a weakly sequentially continuous, strictly convex functional. Then $\Phi(\zeta)$ attains a maximum relative to $\zeta \in \mathcal{F}(\xi)$, for $\xi \geq a\pi^{1/2}$. Moreover, if $\hat{\zeta}$ is a maximiser and $\psi \in \partial\Phi(\hat{\zeta})$, the subdifferential of Φ at $\hat{\zeta}$, then*

$$\hat{\zeta} = \phi \circ \psi, \quad a.e. \ in \ \Pi_+(\xi),$$

for some increasing function ϕ.

Lemma 6 *For every $\lambda > 0$ and $\xi \geq a\pi^{1/2}$, the problem $P_\lambda(\xi)$ is solvable. Moreover, if $\zeta \in \Sigma_\lambda(\xi)$, then*

$$\zeta = \phi \circ (K_+\zeta - \lambda x_1 x_2), \quad a.e \ in \, \Pi_+, \tag{5.6}$$

for some increasing function ϕ.

Proof. Let us begin by noting that $K_+ : L^p(\Pi_+(\xi)) \to L^{p^*}(\Pi_+(\xi))$ is a symmetric operator, that is,

$$\int_{\Pi_+} v\, K_+ w = \int_{\Pi_+} w\, K_+ v, \ \ \forall v, w \in L^p(\Pi_+),$$

which readily follows from the symmetry of G_+. Since K_+ is compact, strictly positive and symmetric it follows that Ψ_λ, defined on the set of functions in $L^p(\Pi_+)$ vanishimg outside $\Pi_+(\xi)$, is strictly convex and weakly sequentially continuous. Now by applying Lemma 5 we deduce that $P_\lambda(\xi)$ is solvable. Next we show that if $\zeta \in \Sigma_\lambda(\xi)$, then $K_+\zeta - \lambda x_1 x_2 \in \partial\Psi_\lambda(\zeta)$. For this purpose we consider $\overline{\zeta} \in L^p(\Pi_+)$ which vanishes outside $\Pi_+(\xi)$, then we need to show

$$\Psi_\lambda(\overline{\zeta}) \geq \Psi_\lambda(\zeta) + \int_{\Pi_+} (\overline{\zeta} - \zeta)(K_+\zeta - \lambda x_1 x_2),$$

or equivalently

$$\int_{\Pi_+} (\overline{\zeta} - \zeta)\, K_+(\overline{\zeta} - \zeta) \geq 0,$$

but this is true since K_+ is strictly positive. Therefore, again by Lemma 5, existence of an increasing function ϕ is ensured so that (5.6) holds.$\diamondsuit$

5.4 Main results and proof of the theorem

In this section we give a proof of Theorem 1 which takes the form of three lemmas and a corollary.

Lemma 7 *Let $\lambda > 0$. Then there exists $R(\lambda) > 0$ such that*

$$K_+\zeta(x) - \lambda x_1 x_2 \leq 0, \ \ |x| \geq R(\lambda), \ \ \zeta \in \mathcal{F}.$$

Proof. Let us fix $x \in \Pi_+$ and $\zeta \in \mathcal{F}$; we assume $\min\{x_1, x_2\} \geq \alpha$ for some $\alpha > 0$ to be determined later. According to Lemma 1(ii) there exists $M > 0$, independent of ζ, such that $|\nabla K_+\zeta(x)| \leq M$. Therefore, by Lemma 1(iii) we have $|K_+\zeta(x)| \leq M \ min\{x_1, x_2\}$. Hence

$$K_+\zeta(x) - \lambda x_1 x_2 \leq min\{x_1, x_2\} \ (M - \lambda\alpha).$$

Thus if we assume $\alpha \geq M/\lambda$, then $K_+\zeta(x) - \lambda x_1 x_2 \leq 0$. Hence we can take $R(\lambda) = M/\lambda$.$\diamondsuit$

Lemma 8 *Suppose $\zeta \in L^p(\mathbb{R}^2)$ is a non-negative function which is spherically decreasing and vanishes outside B_a. Then there exists a positive constant k such that*

$$\int_{\Pi_+} \zeta_t K_+ \zeta_t \geq k \log t,$$

for all sufficiently large t, where $\zeta_t(x) := \zeta(x_1 - t, x_2 - t)$.

Proof. Clearly we can assume $t \geq (1 + \sqrt{2})a$. Now we observe that there exists $\beta > 0$ and $0 < b < a$ such that for all x with $|x| \leq b$ we have $\zeta(x) \geq \beta$. Let $B_b(t)$ denote the ball centred at (t, t) with radius b and consider $x \in B_a(t)$ and $y \in B_b(t)$. Hence if we set

$$\gamma_1 := |x - \overline{y}|, \ \ \gamma_2 := |x - \underline{y}|, \ \ \gamma_3 := |x - y|, \ \ \gamma_4 := |x - \overline{\underline{y}}|,$$

then it is clear that

$$\gamma_1 \geq 2t - 2a, \ \ \gamma_2 \geq 2t - 2a, \ \ \gamma_3 \leq 2a, \ \ \gamma_4 \leq 2\sqrt{2}t + 2a.$$

Therefore,

$$K_+\zeta_t(x) \geq \frac{\beta}{2\pi} \int_{B_b(t)} \log \frac{(2t - 2a)^2}{2a(2\sqrt{2}t + 2a)} \, dy = \frac{\beta b^2}{2} \log \frac{(t - a)^2}{a(\sqrt{2}t + a)}.$$

Hence

$$\int_{\Pi_+} \zeta_t K_+ \zeta_t \geq \frac{\pi \beta^2 b^4}{2} \log \frac{(t - a)^2}{a(\sqrt{2}t + a)},$$

and we are done.$\diamondsuit$

An immediate consequence of Lemma 8 is the following

Corollary *We have*

$$lim_{\xi\to\infty} \sup_{\zeta\in\mathcal{F}(\xi)} \Psi(\zeta) = +\infty.$$

Proof Let $\zeta_1 \in \mathcal{F}$ and denote its Schwarz-rearrangement by ζ^*. Then by Lemma 8 we have

$$\sup_{\zeta\in\mathcal{F}(\xi)} \Psi(\zeta) \geq 1/2 \int_{\Pi_+} \zeta_t^* K_+ \zeta_t^* \to \infty, \cdot\ \xi \to \infty.$$

Hence we are done. $\diamondsuit$

Lemma 9 *There exists $\lambda_0 > 0$ and $\xi_0 > a\pi^{1/2}$ such that if $0 < \lambda \leq \lambda_0$, $\xi \geq \xi_0$ and $\zeta_{\lambda,\xi}$ is a maximiser of $\Psi_\lambda(\zeta)$ relative to $\zeta \in \mathcal{F}(\xi)$ then*

$$|\{x \in \Pi_+(\xi) \mid K_+\zeta_{\lambda,\xi}(x) - \lambda x_1 x_2 > 0\}| \geq \pi a^2.$$

Proof Let us fix $\alpha > 0$, $\epsilon > 0$. Then according to the corollary there exists $\xi_0 \geq a\pi^{1/2}$ such that $\sup_{\zeta\in\mathcal{F}(\xi)} \Psi(\zeta) \geq \alpha + \epsilon$. In particular, $\sup_{\zeta\in\mathcal{F}(\xi_0)} \Psi(\zeta) \geq \alpha + \epsilon$. Since Ψ is a real-valued functional on $L^p(\Pi_+(\xi_0))$ which is weakly sequentially continuous and strictly convex we can apply Lemma 5 to ensure existence of $\hat{\zeta} \in \mathcal{F}(\xi_0)$ such that $\Psi(\hat{\zeta}) = \sup_{\zeta\in\mathcal{F}(\xi_0)} \Psi(\zeta)$, hence

$$\Psi(\hat{\zeta}) \geq \alpha + \epsilon. \tag{5.7}$$

Now choose $\lambda_0 > 0$ such that $\lambda_0 \Im(\hat{\zeta}) < \epsilon$. Since $\Psi_\lambda(\hat{\zeta}) := \Psi(\hat{\zeta}) - \lambda\Im(\hat{\zeta})$ we can use (5.7) to obtain

$$\Psi_\lambda(\hat{\zeta}) \geq \alpha, \ \ 0 < \lambda \leq \lambda_0.$$

This shows that

$$\sup_{\zeta\in\mathcal{F}(\xi)} \Psi_\lambda(\zeta) \geq \alpha, \ \ 0 < \lambda \leq \lambda_0, \ \xi \geq \xi_0. \tag{5.8}$$

Next we set $\alpha = 3aC\|\zeta_0\|_p \ \|\zeta_0\|_1$, where C is the constant in Lemma 1(iii). Hence from (5.8) we have

$$\sup_{\zeta\in\mathcal{F}(\xi)} \Psi_\lambda(\zeta) \geq 3aC\|\zeta_0\|_p \ \|\zeta_0\|_1,$$

for all $0 < \lambda \leq \lambda_0$ and $\xi \geq \xi_0$. Now we fix $0 < \lambda \leq \lambda_0$, $\xi \geq \xi_0$ and let $\zeta_{\lambda,\xi}$ denote a maximiser of $\Psi_\lambda(\zeta)$ relative to $\zeta \in \mathcal{F}(\xi)$. Then we have

$$\Psi_\lambda(\zeta_{\lambda,\xi}) \leq \|\zeta_0\|_1 \sup_{\Pi_+(\xi)} (\frac{1}{2}K_+\zeta_{\lambda,\xi}(x) - \lambda x_1 x_2).$$

We also have $\Psi_\lambda(\zeta_{\lambda,\xi}) \geq 3aC\|\zeta_0\|_p\ \|\zeta_0\|_1$, hence

$$\sup_{\Pi_+(\xi)} (\frac{1}{2}K_+\zeta_{\lambda,\xi}(x) - \lambda x_1 x_2) > 3aC\|\zeta_0\|_p. \tag{5.9}$$

Since $\frac{1}{2}K_+\zeta_{\lambda,\xi}(x) - \lambda x_1 x_2 \in C(\overline{\Pi_+(\xi)})$, it attains its maximum at $(x_1^0, x_2^0) \in \overline{\Pi_+(\xi)}$, say. Whence, by Lemma 1(iii)

$$\sup_{\Pi_+(\xi)} (\frac{1}{2}K_+\zeta_{\lambda,\xi}(x) - \lambda x_1 x_2) \leq \frac{1}{2}K_+\zeta_{\lambda,\xi}(x_1^0, x_2^0) \leq \frac{C}{2}\min\{x_1^0, x_2^0\}\ \|\zeta_0\|_p.$$

Therefore, from (5.9) we infer $\min\{x_1^0, x_2^0\} \geq 6a > 2a$. Now we define the set

$$S := \{x \in \Pi_+ \mid x_1 < x_1^0,\ x_2 < x_2^0\ \} \cap B_{2a}(x_1^0, x_2^0),$$

where $B_{2a}(x_1^0, x_2^0)$ denotes the ball centred at (x_1^0, x_2^0) with radius $2a$; clearly $S \subset \overline{\Pi}_+(\xi)$. Consider $x \in S$, then

$$K_+\zeta_{\lambda,\xi}(x) - \lambda x_1 x_2 \geq 1/2K_+\zeta_{\lambda,\xi}(x) - \lambda x_1^0 x_2^0. \tag{5.10}$$

On the other hand, by an application of the Mean Value Theorem and Lemma 1(ii),

$$\begin{aligned} |K_+\zeta_{\lambda,\xi}(x) - K_+\zeta_{\lambda,\xi}(x_1^0, x_2^0)| &\leq |\nabla K_+\zeta_{\lambda,\xi}(\hat{x})||x - (x_1^0, x_2^0)| \\ &\leq 2aC\|\zeta_0\|_p, \end{aligned}$$

where $\hat{x}$ is a point on the segment joining x to (x_1^0, x_2^0), whence $K_+\zeta_{\lambda,\xi}(x) \geq K_+\zeta_{\lambda,\xi}(x_1^0, x_2^0) - 2aC\|\zeta_0\|_p$. This, in turn, implies

$$K_+\zeta_{\lambda,\xi}(x) \geq K_+\zeta_{\lambda,\xi}(x_1^0, x_2^0) - 2aC\|\zeta_0\|_p. \tag{5.11}$$

Thus from (5.10) and (5.11) we infer

$$\begin{aligned} K_+\zeta_{\lambda,\xi}(x) - \lambda x_1 x_2 &\geq \frac{1}{2}K_+\zeta_{\lambda,\xi}(x_1^0, x_2^0) - aC\|\zeta_0\|_p - \lambda x_1^0 x_2^0 \\ &\geq 3aC\|\zeta_0\|_p - aC\|\zeta_0\|_p = 2aC\|\zeta_0\|_p. \end{aligned}$$

Therefore, $S \subseteq \mathrm{supp}(K_+\zeta_{\lambda,\xi}(x) - \lambda x_1 x_2)$. Hence, $|\mathrm{supp}(K_+\zeta_{\lambda,\xi}(x) - \lambda x_1 x_2)| \geq |S| = \pi a^2$, as desired.$\diamondsuit$

Proof of Theorem 1 Let ξ_0 and λ_0 be as in Lemma 9. If we fix $\lambda \in (0, \lambda_0)$, then by

Lemma 7 there exists $R(\lambda) > 0$ such that

$$K_+\zeta(x) - \lambda x_1 x_2 \leq 0, \;\; x \in \Pi_+ \setminus \Pi_+(R(\lambda)), \;\; \zeta \in \mathcal{F}. \tag{5.12}$$

Next we define $\xi^* := \max\{\xi_0, R(\lambda)\}$; then according to Lemma 6, $\Psi_\lambda(\zeta)$ has a maximiser relative to $\zeta \in \mathcal{F}(\xi^*)$, say ζ_{λ,ξ^*}. For simplicity we write $\hat{\zeta} := \zeta_{\lambda,\xi^*}$. We claim that $\hat{\zeta} \in \Sigma_\lambda$. To prove this , suppose $l \geq \xi^*$ and consider $\overline{\zeta} \in \Sigma_\lambda(l)$. we will first show that

$$\mathrm{supp}(\overline{\zeta}) \subseteq \Pi_+(\xi^*), \tag{5.13}$$

modulo a set of measure zero. By Lemma 6 there exists an increasing function $\overline{\phi}$ such that

$$\overline{\zeta} = \overline{\phi} \circ (K_+\overline{\zeta} - \lambda x_1 x_2), \;\; \text{a.e. in } \Pi_+(l). \tag{5.14}$$

Next we observe that since $\overline{\phi}$ is increasing, $(\overline{\phi})^{-1}(0,\infty]$, the pre-image of $(0,\infty]$ under $\overline{\phi}$, is an interval, say I, of the form (c,∞) or $[c,\infty)$, by assuming that $\overline{\phi}$ takes on the value $+\infty$ on the interval $(\|K_+\overline{\zeta} - \lambda x_1 x_2\|_{\infty,\overline{\Pi_+(l)}}, \infty)$. This, along with (5.14), implies

$$\mathrm{supp}(\overline{\zeta}) = (K_+\overline{\zeta} - \lambda x_1 x_2)^{-1}(I),$$

modulo a set of measure zero in $\Pi_+(l)$. Hence $|(K_+\overline{\zeta} - \lambda x_1 x_2)^{-1}(I)| = \pi a^2$. On the other hand, from Lemma 9, we have $|(K_+\overline{\zeta} - \lambda x_1 x_2)^{-1}(0,\infty)| \geq \pi a^2$. Whence $c \geq 0$, and this implies $\mathrm{supp}(\overline{\zeta}) \subseteq \mathrm{supp}(K_+\overline{\zeta} - \lambda x_1 x_2)$ modulo a set of measure zero in $\Pi_+(l)$. Finally, according to (5.12), we also have $\mathrm{supp}(K_+\overline{\zeta} - \lambda x_1 x_2) \subseteq \Pi_+(\xi^*)$, hence we derive (5.13). Now from (5.13) we infer $\Psi_\lambda(\hat{\zeta}) \geq \Psi_\lambda(\overline{\zeta})$ and this, in turn, implies that $\hat{\zeta} \in \Sigma_\lambda(l)$. Since $l \geq \xi^*$ is arbitrary we deduce that $\hat{\zeta} \in \Sigma_\lambda$.

To derive (5.2) we use Lemma 6 once again to ensure existence of an increasing function $\hat{\phi}$ such that

$$\hat{\zeta} = \hat{\phi} \circ (K_+\hat{\zeta} - \lambda x_1 x_2), \;\; \text{a.e. in } \;\Pi_+(\xi^*).$$

We obtain (5.2) by a *modification process*, that is, define

$$\phi(s) := \begin{cases} \hat{\phi}(s), & s \in \mathrm{dom}(\hat{\phi}),\; s > 0, \\ 0, & s \leq 0. \end{cases}$$

Therefore, clearly, we have

$$\hat{\zeta} = \phi \circ (K_+\hat{\zeta} - \lambda x_1 x_2), \;\; \text{a.e. in } \Pi_+,$$

as required. $\diamondsuit$

Lewt us conclude with the following

Remark A close inspection of the proofs of Theorem 1 and Lemma 9 confirms that if $\zeta \in \Sigma_\lambda$, then $\mathrm{supp}(\zeta) \subseteq \{x \in \Pi_+ \mid K_+\zeta(x) - \lambda x_1 x_2 \geq 2aC\|\zeta_0\|_p\}$, modulo a set of measure zero. Hence, for almost every $x \in \mathrm{supp}(\zeta)$ we have

$$\begin{aligned} 2aC\|\zeta_0\|_p \leq K_+\zeta(x) - \lambda x_1 x_2 &\leq K_+\zeta(x) \\ &\leq C\min\{x_1, x_2\}\|\zeta_0\|_p, \end{aligned}$$

where in the last inequality we have used Lemma 1(iii). Therefore, for almost every $x \in \mathrm{supp}(\zeta)$

$$\min\{x_1, x_2\} \geq 2a.$$

This shows that the vortex core essentially avoids the boundary of Π_+.

Chapter 6

Existence of a steady flow with a bounded vortex in an unbounded domain (part two)

6.1 Introduction

This chapter is concerned with the same problem discussed in Chapter 5 and again motivated by [12]. Here we use a different variational approach. We assume the vorticity field ζ is a rearrangement of a prescribed non-negative function ζ_0 and the ' impulse ', $\Im$, given by

$$\Im(\zeta) := \int_{\Pi_+} x_1 x_2 \zeta$$

is a prescribed positive number. In our result we prove the variational problem, $P(I)$, is solvable provided I is sufficiently large. Since the domain of interest, Π_+, is unbounded we first consider the problem over bounded sets, $\Pi_+(\xi, \eta)$, where Burton's theory, related to constrained variational problems, can be applied. We then show that the maximisers are the same for all sufficiently large $\Pi_+(\xi, \eta)$. Our analysis finally leads to the existence of strong solutions to the following semilinear elliptic partial differential equation

$$-\Delta\psi = \phi \circ (\psi - \lambda x_1 x_2), \text{ a.e. in } \Pi_+$$

where ϕ is an increasing function and $\lambda \in \mathbb{R}$ is a Lagrange multiplier, both unknown a priori. It is shown that $\lambda > 0$.

6.2 Notation, definitions and statement of the results

Notation used in Chapter 5 is valid in this chapter as well; with one exception, namely

$$\Pi_+(\eta) := \{x \in \Pi_+ |\ x_1 x_2 < \eta\}.$$

Now we proceed to introduce more notation. For positive constants η and ξ we define

$$\Pi_+(\xi, \eta) := \{x \in \Pi_+ |\ x_1 x_2 < \eta,\ \max\{x_1, x_2\} < \xi\}.$$

The set of functions $\zeta \in \mathcal{F}$ that satisfy $\Im(\zeta) = I$, for some $I > 0$, is denoted $\mathcal{F}(I)$; and the set of functions in $\mathcal{F}(I)$ that vanish outside $\Pi_+(\xi, \eta)$ is denoted $\mathcal{F}(\xi, \eta, I)$; to ensure $\mathcal{F}(\xi, \eta, I) \neq \emptyset$ we present the following definition: Let $I_1 := \Im(\zeta_0^*)$, where ζ_0^* is the Schwarz-symmetrisation of ζ_0, and assume $I > I_1$; we say $\Pi_+(\xi, \eta)$ satisfies the hypothesis $\mathcal{H}(I)$ if the following two conditions hold

$$\xi \geq \eta^{1/2}, \tag{6.1}$$

$$\eta \geq 4\max\{a^2,\ l(I)\}, \tag{6.2}$$

where $l(I) := (I - I_1)/||\zeta_0||_1$. Now it is immediate that if $\Pi_+(\xi, \eta)$ satisfies $\mathcal{H}(I)$, for $I > I_1$, then $\mathcal{F}(\xi, \eta, I) \neq \emptyset$. Indeed if we set $t = l(I)^{1/2}$, then $(\zeta_0^*)_t(x) := \zeta_0^*(x_1 - t, x_2 - t)$ belongs to $\mathcal{F}(\xi, \eta, I)$.

In this chapter we are concerned with constained variational problems which are defined as follows. For $I > I_1$,

$$P(I): \quad \sup_{\zeta \in \mathcal{F}(I)} \Psi(\zeta);$$

and the coresponding solution set is denoted $\Sigma(I)$. If $I > I_1$ and $\Pi_+(\xi, \eta)$ satisfies $\mathcal{H}(I)$, then we define the truncated variational problem

$$P(\xi, \eta, I): \quad \sup_{\zeta \in \mathcal{F}(\xi, \eta, I)} \Psi(\zeta),$$

with the solution set $\Sigma(\xi, \eta, I)$.

We are now in a position to state our main result.

Theorem 1 *There exists $I_0 > 0$ such that if $I > I_0$ then $P(I)$ has a solution, that is, $\Sigma(I) \neq \emptyset$; if ζ is a solution and $\psi := K_+\zeta$ then the following semilinear elliptic partial* *differential equation holds*

$$-\Delta\psi = \phi \circ (\psi - \lambda x_1 x_2), \quad a.e.\ in\, \Pi_+, \tag{6.3}$$

where ϕ is an increasing function and $\lambda > 0$, both unknown a priori. Furthermore, I_0 can be chosen to ensure the vortex core, the strong support of ζ, avoids $\partial\Pi_+$.

6.3 Preliminary results

The proof of Theorem 1 is based on a number of lemmas which will be the content of this section. We begin by stating a lemma from Burton's theory, see for example [14].

Lemma 1 *Let Ω be a non-empty open set in $\mathbb{R}^n$. Let $1 \le p < \infty$ and p^* denote the conjugate exponent of p. For $\zeta \in L^p(\mu)$ let $\mathcal{F}(\Omega)$ denote the set of rearrangements of ζ on Ω. Let*

$$\mathcal{L} := \sum_{1 \le |\alpha| \le m} \mathcal{A}^\alpha(x)\mathcal{D}^\alpha$$

be an m^{th}-order linear partial differential operator, whose coeficients $\mathcal{A}^\alpha$ are finite-valued measurable functions on Ω, having no 0^{th}-order term, and suppose there exists a compact, symmetric, positive linear operator $K : L^p(\Omega) \to L^{p^*}(\Omega)$ such that if $\zeta \in L^p(\Omega)$, then $K\zeta \in L^{p^*}(\Omega) \cap W^{m,1}_{loc}(\Omega)$ and $\mathcal{L}K\zeta = \zeta$ almost everywhere in Ω. Define

$$\hat{\Psi}(\zeta) := \int_\Omega \zeta K\zeta, \quad \zeta \in L^p(\Omega).$$

Let $w \in L^{p^*}(\Omega) \cap W^{m,1}_{loc}(\Omega)$ be such that $\mathcal{L}w$ is essentially constant, and define

$$\mathcal{T}(\zeta) := \int_\Omega w\zeta, \quad \zeta \in L^p(\Omega).$$

Let $b \in \mathbb{R}$. Then
(i) If $b \in \mathcal{T}(\mathcal{F}(\Omega))$ then

$$\sup \hat{\Psi}(\mathcal{T}^{-1}(b) \cap \mathcal{F}(\Omega)) = \sup \hat{\Psi}(\mathcal{T}^{-1}(b) \cap \overline{\mathcal{F}(\Omega)^w}),$$

and the supremum is attained by at least one element of $\mathcal{T}^{-1}(b) \cap \mathcal{F}(\Omega)$.
(ii) If b is relatively interior to $\mathcal{T}(\mathcal{F}(\Omega))$, and if $\overline{\zeta}$ is a maximiser for Ψ relative to $\mathcal{T}^{-1}(b) \cap \mathcal{F}(\Omega)$, then there exist scalar λ and an increasing function ϕ such that

$$\overline{\zeta} = \phi \circ (K\overline{\zeta} + \lambda w), \quad a.e.\ in\, \Omega.$$

Remark It is clear that if $I > I_1$ and $\Pi_+(\xi, \eta)$ satisfies $\mathcal{H}(I)$ then, by Lemma 1(i), $\Sigma(\xi, \eta, I) \neq \emptyset$.

Before stating the next result we give the following definition: For $I > I_1$,

$$\sigma(I) := \inf\{\Psi(\zeta) | \ \zeta \in \Sigma(\xi, \eta, I), \text{ for some } \Pi_+(\xi, \eta) \text{ satisfying } \mathcal{H}(I)\}.$$

Let us point out that $\sigma(I) = \Psi(\hat{\zeta})$ for some $\hat{\zeta} \in \Sigma(\xi_0, \eta_0, I)$, where $\Pi_+(\xi_0, \eta_0)$ is the minimal region that satisfies $\mathcal{H}(I)$.

Lemma 2

$$\lim_{I \to \infty} \sigma(I) = \infty.$$

Proof Let $I > I_1$ and set $t = l(I)^{1/2}$. If $\Pi_+(\xi, \eta)$ satisfies $\mathcal{H}(I)$, then $(\zeta_0^*)_t \in \mathcal{F}(\xi, \eta, I)$ and therefore, according to the last remark, we have

$$\sigma(I) \geq \Psi((\zeta_0^*)_t).$$

Now applying [Chapter 5, Lemma 8], we obtain $\Psi((\zeta_0^*)_t) \geq k \log t$, for all sufficiently large t, hence large I. Thus we are done.$\diamondsuit$

Let $I > I_1$ and $\Pi_+(\xi, \eta)$ satisfies $\mathcal{H}(I)$. We set

$$M(\xi, \eta, I) := \left\{ (\zeta, \phi, \lambda) \left| \begin{array}{c} \zeta \in \Sigma(\xi, \eta, I) \text{ for some } \phi, \ \lambda \in \mathbb{R} \\ \text{such that } \zeta = \phi \circ (K_+ - \lambda x_1 x_2) \text{ a.e. in } \Pi_+(\xi, \eta) \end{array} \right. \right\}.$$

Note that under the conditions imposed on ξ, η, I and in view of Lemma 1(ii) the set $M(\xi, \eta, I)$ is non-empty. Prior to presenting the next result we recall from [Chapter 5, Lemma 1] that there exists $N > 0$ such that

$$|K_+\zeta(x)| \leq N \min\{x_1, x_2\}, \tag{6.4}$$

$$|\nabla K_+\zeta(x)| \leq N, \tag{6.5}$$

for every $x \in \Pi_+$ and every $\zeta \in \mathcal{F}$.

Lemma 3 *For $I > I_1$ we define*

$$\Lambda(I) := \sup \left\{ \lambda \left| \begin{array}{c} (\zeta, \phi, \lambda) \in M(\xi, \eta, I) \ \textit{for some } \zeta, \phi \textit{ and some } \Pi_+(\xi, \eta) \\ \textit{satisfying } \mathcal{H}(I) \end{array} \right. \right\}.$$

Then, $\limsup_{I \to \infty} \Lambda(I) \leq 0$.

Proof Let us assume the assertion of the lemma is not true and seek a contradiction. Hence to this end we suppose there exists $\beta \in (0, \infty]$ such that $\limsup_{I \to \infty} \Lambda(I) = \beta$. Hence there exists $\Lambda > 0$ such that the set

$$S := \{I | \, \Lambda(I) > \Lambda\}$$

is unbounded. Consider $I \in S$, then from the definition of $\Lambda(I)$ there exists $(\zeta, \phi, \lambda) \in M(\xi, \eta, I)$ such that $\Pi_+(\xi, \eta)$ satisfies $\mathcal{H}(I)$ and $\Lambda(I) \geq \lambda > \Lambda > 0$. Observe that by taking I sufficiently large we can ensure the existence of ξ_1 such that $\Pi_+(\xi, \eta) \supseteq \Pi_+(\xi_1, a)$ and $|\Pi_+(\xi_1, a)| \geq \pi a^2 = |\text{supp}(\zeta)|$. Now define the set

$$U := \{x \in \Pi_+(\xi, \eta) | \, K_+\zeta(x) - \lambda x_1 x_2 \geq -\lambda a\}.$$

Then, $\Pi_+(\xi_1, a) \subseteq U$ and $|U| \geq |\text{supp}(\zeta)|$. Since ζ is essentially an increasing function of $K_+\zeta - \lambda x_1 x_2$ on $\Pi_+(\xi, \eta)$ we deduce that $\text{supp}(\zeta) \subseteq U$.

Next we show that there exists a constant $C > 0$, independent of $I \in S$, such that for $x \in \text{supp}(\zeta)$ we have $x_1 x_2 \leq C$. From (6.4) we observe that for a sufficiently large $k > 0$

$$K_+\zeta(x) \leq \frac{\Lambda}{2} x_1 x_2, \tag{6.6}$$

for all $\zeta \in \mathcal{F}$ and all x for which $\min\{x_1, x_2\} \geq k$. We next define

$$\begin{aligned} S_1 &:= \{x \in \Pi_+ | \, \min\{x_1, x_2\} \geq k\}, \\ S_2 &:= \{x \in \Pi_+ | \, \min\{x_1, x_2\} < k, \; x_1 < \alpha, \; x_2 < \alpha\}, \\ S_3 &:= \{x \in \Pi_+ | \, \min\{x_1, x_2\} < k, \; \max\{x_1, x_2\} \geq \alpha\}, \end{aligned}$$

where $\alpha := \max\{2N/\lambda, k\}$. First consider $x \in \text{supp}(\zeta) \cap S_1$; then

$$\begin{aligned} -\lambda a &\leq K_+\zeta(x) - \lambda x_1 x_2 \\ &\leq \frac{\Lambda}{2} x_1 x_2 - \lambda x_1 x_2 < -\frac{\lambda}{2} x_1 x_2, \end{aligned}$$

where the first inequality follows from $\text{supp}(\zeta) \subseteq U$ and the second one from (6.6); whence $x_1 x_2 < 2a$. Next consider $x \in \text{supp}(\zeta) \cap S_2$; then we have

$$x_1 x_2 < \alpha^2 \leq (\max\{2N/\Lambda, k\})^2,$$

since $\lambda > \Lambda$. Finally, consider $x \in \text{supp}(\zeta) \cap S_3$; then an application of (6.4) yields

$$-\lambda a \leq K_+\zeta(x) - \lambda x_1 x_2 \;\; \leq \;\; N \min\{x_1, x_2\} - \lambda x_1 x_2$$

$$\begin{aligned}
&= \frac{N}{\alpha}\alpha\min\{x_1,x_2\} - \lambda x_1x_2 \\
&\leq \frac{N}{\alpha}x_1x_2 - \lambda x_1x_2 \\
&\leq N\frac{\lambda}{2N}x_1x_2 - \lambda x_1x_2 = -\frac{\lambda}{2}x_1x_2,
\end{aligned}$$

hence $x_1x_2 \leq 2a$. Therefore, from above argument, it is clear that a constant $C > 0$, as required, exists. This, in turn, implies that

$$I = \Im(\zeta) := \int_{\Pi_+} x_1x_2\zeta \leq C\|\zeta_0\|_1.$$

Thus S is bounded, which is a contradiction. Hence we are done.$\diamondsuit$

Lemma 4 *For $I > I_1$ we define*

$$A(I) := \inf\left\{ \textit{ess inf}_{x\in supp(\zeta)}(K_+\zeta(x) - \lambda x_1x_2) \;\middle|\; \begin{array}{c} (\zeta,\phi,\lambda)\in M(\xi,\eta,I) \\ \textit{for some } \Pi_+(\xi,\eta) \\ \textit{and some } \phi \end{array} \right\},$$

where $\Pi_+(\xi,\eta)$ is to satisfy $\mathcal{H}(I)$. Then, $\liminf_{I\to\infty} A(I) \geq 0$.

Proof Fix $\epsilon > 0$. By definition of $A(I)$ there exists $\Pi_+(\xi,\eta)$, satisfying $\mathcal{H}(I)$, and $(\zeta,\phi,\lambda)\in M(\xi,\eta,I)$ such that

$$A(I) + \epsilon \geq \text{ess inf}_{x\in\text{supp}(\zeta)}(K_+\zeta(x) - \lambda x_1x_2). \tag{6.7}$$

Note that by increasing I the size of $\Pi_+(\xi,\eta)$ increases as well, hence there is no loss of generality if we assume $\Pi_+(\xi,\eta)$ contains the square $D := [0,2a]\times[0,2a]$, since I will eventually tend to infinity. For $x \in D$ we have

$$K_+\zeta(x) - \lambda x_1x_2 \geq -4a^2\Lambda(I)^+,$$

where $\Lambda(I)^+$ denotes the positive part of $\Lambda(I)$, since $K_+\zeta$ is non-negative. From this we infer

$$D \subseteq \{x \in \Pi_+(\xi,\eta) |\, K_+\zeta(x) - \lambda x_1x_2 \geq -4a^2\Lambda(I)^+\}.$$

Hence

$$|\{x \in \Pi_+(\xi,\eta) |\, K_+\zeta(x) - \lambda x_1x_2 \geq -4a^2\Lambda(I)^+\}| > |\text{supp}(\zeta)|,$$

since $4a^2 > \text{supp}(\zeta)$. Since ζ is essentially an increasing function of $K_+\zeta - \lambda x_1 x_2$ on $\Pi_+(\xi, \eta)$ we then deduce

$$\text{supp}(\zeta) \subseteq \{x \in \Pi_+(\xi,\eta) |\, K_+\zeta(x) - \lambda x_1 x_2 \geq -4a^2 \Lambda(I)^+\},$$

hence by applying (6.7) we obtain $A(I) + \epsilon \geq -4a^2\Lambda(I)^+$. Therefore from Lemma 3 we have

$$\liminf_{I \to \infty} A(I) + \epsilon \geq 0.$$

Since ϵ was arbitrary we derive the desired conclusion.$\diamondsuit$

We now present an auxiliary lemma which is used to prove Lemma 6(see below). Lemma 5 is proved similarly to [12, Lemma 8] except that here we give a direct proof without using a density argument. Lemmas 5 and 6 bear some resemblance to Pohazaev-type identities proved in [26] for 3-dimensional vortex rings.

Lemma 5 *Let $2 < p < \infty$, let $\zeta \in L^p(\Pi_+)$ have bounded support and let $\psi := K_+\zeta$. Then*

$$\int_{\Pi_+} (x \cdot \nabla\psi)\zeta = 0. \tag{6.8}$$

Proof Let $\theta := |\nabla\psi|^2$ and $\sigma := |x|^2/4$. Since $\psi \in C^1(\mathbb{R}^2) \cap W^{2,p}_{\text{loc}}(\mathbb{R}^2)$ and $\Pi_+ \cap B_R$, $R > 0$, is a Lipschitz domain, we can apply the Weak Divergence Theorem, see for example [29], to obtain

$$\int_{\Pi_+ \cap B_R} \theta = \int_{\Pi_+ \cap B_R} \theta\Delta\sigma = \int_{\Pi_+ \cap \partial B_R} \theta \nabla\sigma \cdot \hat{x} - \int_{\Pi_+ \cap B_R} \nabla\theta \cdot \nabla\sigma,$$

where $\hat{x}$ is the unit vector in direction x, since $\nabla\sigma$ is tangential on $\partial\Pi_+$. Since $\nabla\psi = O(|x|^{-2})$ as $|x| \to \infty$, see[Chapter 4, Lemma 3], we can apply the Lebesgue Dominated Convergence Theorem to obtain

$$\int_{\Pi_+} \theta = -\lim_{R \to \infty} \int_{\Pi_+ \cap B_R} \nabla\theta \cdot \nabla\sigma.$$

The following identity was derived in [12, Lemma 8]:

$$\nabla\theta \cdot \nabla\sigma = \nabla\psi \cdot \nabla(x \cdot \nabla\psi).$$

Thus, since $x \cdot \nabla\psi \in W^{1,2}(\Pi_+ \cap B_R)$, another application of the Weak Divergence Theorem yields

$$\int_{\Pi_+ \cap B_R} \nabla\theta \cdot \nabla\sigma \quad = \quad \int_{\Pi_+ \cap B_R} \nabla\psi \cdot \nabla(x \cdot \nabla\psi)$$

$$= -\int_{\Pi_+\cap B_R}((x\cdot\nabla\psi)\Delta\psi+\theta)+\int_{\partial(\Pi_+\cap B_R)}(x\cdot\nabla\psi)\nabla\psi\cdot\hat{x}.$$

Since $\nabla^\perp\psi\cdot\hat{j}=0$, $\hat{j}$ being the standard unit vector in $\mathbb{R}^2$ perpendicular to the x_1-axis, we infer

$$\int_{\partial(\Pi_+\cap B_R)}(x\cdot\nabla\psi)\nabla\psi\cdot\hat{x}=\int_{\Pi_+\cap\partial B_R}(x\cdot\nabla\psi)\nabla\psi\cdot\hat{x}.$$

Moreover, since $\nabla\psi=O(|x|^{-2})$ as $|x|\to\infty$, we have

$$\lim_{R\to\infty}\int_{\Pi_+\cap\partial B_R}(x\cdot\nabla\psi)\nabla\psi\cdot\hat{x}=0.$$

Therefore, from the Lebesgue Dominated Convergence Theorem, we obtain

$$\int_{\Pi_+}\theta=\int_{\pi_+}(x\cdot\nabla\psi)\Delta\psi+\int_{\Pi_+}\theta,$$

from which (6.8) follows.$\Diamond$

The next lemma can be proved using the same argument as in [12, Lemma 9].

Lemma 6 *Let $2<p<\infty$, let $\zeta\in L^p(\Pi_+)$ be non-negative, non-trivial and vanish outside the square $D(\xi):=[0,\xi]\times[0,\xi]$, for some $\xi>0$. Let $\lambda\in\mathbb{R}$, and let $\psi:=K_+\zeta-\lambda x_1x_2$. Suppose $\zeta=\phi\circ\psi$ almost everywhere in $D(\xi)$ for some increasing function ϕ, and suppose ϕ has a non-negative indefinite integral F. Then*

$$2\int_{D(\xi)}F\circ\psi-2\lambda\int_{D(\xi)}x_1x_2\zeta=\int_{\partial D(\xi)}(F\circ\psi)(x\cdot\vec{n}),$$

where $\vec{n}$ is the outward unit normal, and consequently

$$\int_{D(\xi)}F\circ\psi\geq\lambda\int_{D(\xi)}x_1x_2\zeta.$$

If additionally $F(s)=0$ for some $s\leq\beta$, then

$$\int_{D(\xi)}\zeta K_+\zeta\geq 2\lambda\int_{D(\xi)}x_1x_2\zeta+\beta\|\zeta\|_1.$$

Lemma 7 *For $I>I_1$ we define*

$$\mu(I):=\inf\left\{\sup_{x\in\Pi_+(\xi,\eta)}(K_+\zeta(x)-\lambda x_1x_2)\;\middle|\;\begin{array}{c}(\zeta,\phi,\lambda)\in M(\xi,\eta,I)\\ \text{for some }\Pi_+(\xi,\eta)\text{ satisfying}\\ \mathcal{H}(I)\text{ , and some }\phi\end{array}\right\}.$$

Then $\lim_{I\to\infty}\mu(I)=\infty$.

Proof It clearly suffices to show

$$\liminf_{I\to\infty}\mu(I)=\infty. \tag{6.9}$$

Let $I>I_1$ and consider $(\zeta,\phi,\lambda)\in M(\xi,\eta,I)$ for some $\Pi_+(\xi,\eta)$ satisfying $\mathcal{H}(I)$. Since $K_+\zeta(x)-\lambda x_1x_2\geq\Lambda(I)$ for almost every $x\in\operatorname{supp}(\zeta)$ we may assume that $\phi(s)=0$ for $-\infty<s<A(I)$. Now write

$$F(s)=\int_{-\infty}^{s}\phi,$$

for all s in the domain of ϕ. Now by Lemma 6 we have

$$\begin{aligned}\int_{\Pi_+}\zeta(K_+\zeta-\lambda x_1x_2) &= 2\Psi(\zeta)-\lambda I\\ &= \frac{1}{2}(2\Psi(\zeta)-2\lambda I-A(I)\|\zeta\|_1)\\ &+ \Psi(\zeta)+\frac{1}{2}A(I)\|\zeta\|_1\\ &\geq \Psi(\zeta)+\frac{1}{2}A(I)\|\zeta\|_1\\ &\geq \sigma(I)+\frac{1}{2}A(I)\|\zeta\|_1.\end{aligned}$$

Hence

$$\sup_{\Pi_+(\xi,\eta)}(K_+\zeta(x)-\lambda x_1x_2)\geq\frac{\sigma(I)}{\|\zeta\|_1}+\frac{1}{2}A(I).$$

Therefore

$$\mu(I)\geq\frac{\sigma(I)}{\|\zeta\|_1}+\frac{1}{2}A(I).$$

Thus by applying Lemmas 3 and 4 we obtain (6.9).◊

Lemma 8 *There exists* $I_2>I_1$ *such that*

$$A(I)\geq aN,\ \ I\geq I_2. \tag{6.10}$$

Proof By Lemma 6 there exists $I_2>I_1$ such that

$$\mu(I)\geq 7aN,\ \ I\geq I_2; \tag{6.11}$$

moreover by taking I_2 sufficiently large we can ensure that if $I \geq I_2$, then any $\Pi_+(\xi, \eta)$ satisfying $\mathcal{H}(I)$, also satisfies

$$\left|\Pi_+(\xi,\eta) \setminus \Pi_+(\xi, \frac{\eta}{2})\right| \geq \pi a^2. \tag{6.12}$$

To see this, observe that in general we have

$$|\Pi_+(\xi,\eta)| = \eta\left(1 + \log\frac{\xi^2}{\eta}\right),$$

for any $\Pi_+(\xi,\eta)$ satisfying (6.1); therefore

$$\left|\Pi_+(\xi,\eta) \setminus \Pi_+(\xi, \frac{\eta}{2})\right| \geq \frac{1}{2}(1 - \log 2)\eta.$$

Hence, in view of (6.2), for sufficiently large I we derive (6.12). Now let us fix $I \geq I_2$ and consider $(\zeta, \phi, \lambda) \in M(\xi, \eta, I)$ for some $\Pi_+(\xi,\eta)$ satisfying $\mathcal{H}(I)$. Since $K_+\zeta - \lambda x_1 x_2 \in C(\overline{\Pi_+(\xi,\eta)})$ it attains its maximum at, say, $z \in \overline{\Pi_+(\xi,\eta)}$. Now from the definition of $\mu(I)$ and (6.4) we infer

$$\mu(I) \leq K_+\zeta(z) - \lambda z_1 z_2 \leq N \min\{z_1, z_2\} - \lambda z_1 z_2;$$

and applying (6.11) we obtain

$$7aN \leq N \min\{z_1, z_2\} - \lambda z_1 z_2.$$

Clearly if $\lambda \geq 0$ we obtain $\min\{z_1, z_2\} \geq 7a$. If $\lambda < 0$, then

$$7aN \leq N \min\{z_1, z_2\} - \lambda\eta,$$

or

$$N \min\{z_1, z_2\} \geq 7aN + \lambda\eta.$$

Now we consider two cases:
Case (i): $\lambda\eta \geq -2aN$. Then $N \min\{z_1, z_2\} \geq 5aN$, hence $\min\{z_1, z_2\} \geq 5a$. Therefore when $\lambda \geq 0$ or when $\lambda < 0$ and $\lambda\eta \geq -2aN$ we find $\min\{z_1, z_2\} \geq 5a$. Thus $\Pi_+(\xi,\eta)$ must contain at least a quadrant of $B_{4a}(z)$, denoted by Q. For $x \in Q$, by the Mean Value Inequality, we have

$$\begin{aligned} K_+\zeta(x) - \lambda x_1 x_2 \geq K_+\zeta(x) &\geq K_+\zeta(z) - 4aN \\ &= K_+\zeta(z) - \lambda z_1 z_2 - 4aN + \lambda z_1 z_2 \end{aligned}$$

$$\begin{aligned}
&\geq \mu(I) - 4aN + \lambda z_1 z_2 \geq \mu(I) - 4aN + \lambda\eta \\
&\geq 7aN - 4aN - 2aN = aN.
\end{aligned}$$

This means

$$Q \subseteq \{x \in \Pi_+(\xi,\eta) |\, K_+\zeta(x) - \lambda x_1 x_2 \geq aN\}.$$

Case (ii): $\lambda\eta < -2aN$. Then for $x \in \Pi_+(\xi,\eta) \setminus \Pi_+(\xi,\frac{\eta}{2})$ we have

$$K_+\zeta(x) - \lambda x_1 x_2 \geq -\lambda x_1 x_2 > -\frac{\lambda\eta}{2};$$

$$\Pi_+(\xi,\eta) \setminus \Pi_+(\xi,\frac{\eta}{2}) \subset \{x \in \Pi_+(\xi,\eta) |\, K_+\zeta(x) - \lambda x_1 x_2 \geq aN\}.$$

From (6.12) and the fact that $|Q| = 4\pi a^2$ we infer

$$|\{x \in \Pi_+(\xi,\eta) |\, K_+\zeta(x) - \lambda x_1 x_2 \geq aN\}| \geq |\mathrm{supp}(\zeta)|.$$

Since ζ is an increasing function of $K_+\zeta - \lambda x_1 x_2$ on $\Pi_+(\xi,\eta)$ we derive

$$\mathrm{supp}(\zeta) \subseteq \{x \in \Pi_+(\xi,\eta) |\, K_+\zeta(x) - \lambda x_1 x_2 \geq aN\},$$

modulo a set of zero measure, from which we obtain (6.10).$\diamondsuit$

Lemma 9 *Let $b > 0$, let $2 < p < \infty$ and $0 < \gamma < 1$. Then there exists positive constants M_1, M_2 and M_3 such that*

$$\begin{aligned}
K_+\zeta(x) &\leq M_1(x_1x_2)^{-1}\Im(\zeta) + M_2(x_1x_2)^{-1}\Im(\zeta)\log\frac{25x_1x_2}{4|x|} \\
&+ M_3(x_1x_2)^{-\gamma}\Im(\zeta)^{\gamma}\|\zeta\|_p^{1-\gamma},
\end{aligned} \tag{6.13}$$

for every $x \in \Pi_+$ such that $min\{x_1x_2\} \geq b/2$ and every non-negative $\zeta \in L^p(\Pi_+)$ that vanishes outside a set of measure πb^2.

Proof Fix $x \in \Pi_+$ such that $\nu := \min\{x_1x_2\} \geq b/2$. For $y \in \Pi_+$ we define

$$\alpha := |x - \overline{y}|,\ \beta := |x - \underline{y}|,\ \rho := |x - y|,\ \delta := |x - \underline{\overline{y}}|.$$

Thus

$$\begin{aligned}
K_+\zeta(x) &= \frac{1}{2\pi}\int_{\Pi_+} \log\frac{\alpha\beta}{\rho\delta}\,\zeta(y)dy \\
&= \frac{1}{2\pi}\int_{B_{\nu/2}(x)} \log\frac{\alpha\beta}{\rho\delta}\,\zeta(y)dy + \frac{1}{2\pi}\int_{\Pi_+\setminus B_{\nu/2}(x)} \log\frac{\alpha\beta}{\rho\delta}\,\zeta(y)dy,
\end{aligned}$$

where $B_{\nu/2}(x)$ denotes the ball centered at x with radius ν. From the identity

$$\alpha^2\beta^2 = \rho^2\delta^2 + 16x_1x_2y_1y_2$$

we obtain

$$\begin{aligned}
\int_{\Pi_+\setminus B_{\nu/2}(x)} \log\frac{\alpha\beta}{\rho\delta}\,\zeta(y)dy &= \frac{1}{2}\int_{\Pi_+\setminus B_{\nu/2}(x)} \log(1+\frac{16x_1x_2y_1y_2}{\rho^2\delta^2})\,\zeta(y)dy \\
&\le 8x_1x_2\int_{\Pi_+\setminus B_{\nu/2}(x)} \frac{y_1y_2}{\rho^2\delta^2}\,\zeta(y)dy \\
&\le \frac{32x_1x_2}{\nu^2|x|^2}\int_{\Pi_+\setminus B_{\nu/2}(x)} y_1y_2\,\zeta(y)dy \\
&\le 32(x_1x_2)^{-1}\Im(\zeta), \qquad (6.14)
\end{aligned}$$

where the first inequality follows from the fact that $\log(1+x) \le x$, for $x \ge 0$. To estimate $\int_{B_{\nu/2}(x)} \log(\alpha\beta\rho^{-1}\delta^{-1})\,\zeta(y)dy$ we note that for $y \in B_{\nu/2}(x)$ we have

$$\alpha \le |x-\overline{x}| + |\overline{x}-\overline{y}| = 2x_2 + \rho < \frac{5}{2}x_2.$$

Similarly, $\beta < 5/2x_1$. Therefore

$$\begin{aligned}
\int_{B_{\nu/2}(x)} \log\frac{\alpha\beta}{\rho\delta}\,\zeta(y)dy &\le \int_{B_{\nu/2}(x)} \log\frac{25x_1x_2}{4\rho|x|}\,\zeta(y)dy \\
&= \log\frac{25x_1x_2}{4|x|}\int_{B_{\nu/2}(x)} \zeta(y)dy \\
&+ \int_{B_{\nu/2}(x)} \log\frac{1}{\rho}\,\zeta(y)dy. \qquad (6.15)
\end{aligned}$$

Observe that for $y \in B_{\nu/2}(x)$ we have $y_1y_2 \ge x_1x_2/4$, hence

$$\int_{B_{\nu/2}(x)} \zeta(y)dy \le 4(x_1x_2)^{-1}\int_{B_{\nu/2}(x)} y_1y_2\,\zeta(y)dy \le 4(x_1x_2)^{-1}\Im(\zeta).$$

On the other hand if we let $\hat{\zeta}$ to denote the Schwarz-symmetrisation of $\overline{\zeta} := \zeta\chi_{B_{\nu/2}(x)}$, where $\chi_{B_{\nu/2}(x)}$ is the characteristic function of $B_{\nu/2}(x)$ in Π_+, about x; then by a standard inequality, see for example [5], and Hölder's inequality we obtain

$$\int_{B_{\nu/2}(x)} \log\frac{1}{\rho}\,\zeta(y)dy \le \int_{B_{\nu/2}(x)} \log\frac{1}{\rho}\,\hat{\zeta}(y)dy$$

$$\leq \left(\int_{B_{\hat{b}}(x)} \left| \log \frac{1}{\rho} \right|^{\tau} dy \right)^{\frac{1}{\tau}} \|\hat{\zeta}\|_{\epsilon}, \tag{6.16}$$

where $\hat{b} := |\text{supp}(\zeta \chi_{B_{\nu/2}(x)})|$ $(\leq b)$, $\epsilon := p/(1 + p\gamma - \gamma)$ and τ is the conjugate exponent of ϵ. It is elementary to show that

$$\int_{B_{\hat{b}}(x)} \left| \log \frac{1}{\rho} \right|^{\tau} dy \leq C, \tag{6.17}$$

where C is a constant independent of x. Next observe that $\epsilon = \epsilon\gamma + (1 - \epsilon\gamma)p$ and $\epsilon\gamma < 1$, hence applying the standard interpolation inequality yields

$$\|\hat{\zeta}\|_{\epsilon}^{\epsilon} \leq \|\hat{\zeta}\|_1^{\epsilon\gamma} \, \|\hat{\zeta}\|_p^{(1-\epsilon\gamma)p},$$

or

$$\begin{aligned} \|\hat{\zeta}\|_{\epsilon} &\leq \|\hat{\zeta}\|_1^{\gamma} \, \|\hat{\zeta}\|_p^{(1-\epsilon\gamma)p/\epsilon} \\ &= \|\hat{\zeta}\|_1^{\gamma} \, \|\hat{\zeta}\|_p^{1-\gamma}. \end{aligned}$$

Therefore we obtain

$$\|\hat{\zeta}\|_{\epsilon} \leq 4^{\gamma} (x_1 x_2)^{-\gamma} \Im(\zeta)^{\gamma} \|\zeta\|_p^{1-\gamma}. \tag{6.18}$$

Finally from (6.14)-(6.18) we derive (6.13).$\diamondsuit$

Lemma 10 *Let ζ be a non-negative measurable function on Π_+, let $t > 0$. Let ζ_t be the function, defined on Π_+, obtained by translating ζ along the diagonal of Π_+, diag(Π_+), $\sqrt{2}t$ units, that is,*

$$\zeta_t(x_1, x_2) := \begin{cases} \zeta(x_1 - t, x_2 - t), & x_1 \geq t, \; x_2 \geq t \\ 0, & 0 < x_1 < t, \; 0 < x_2 < t. \end{cases}$$

Then

$$\int_{\Pi_+} \zeta_t K_+ \zeta_t \geq \int_{\Pi_+} \zeta K_+ \zeta. \tag{6.19}$$

Proof Let $\mathbf{t} = (t, t)$. Then

$$\begin{aligned} K_+\zeta_t(x + \mathbf{t}) &= \frac{1}{2\pi} \int_{y_1 \geq t, y_2 \geq t} \log \frac{|x + \mathbf{t} - \overline{y}||x + \mathbf{t} - \underline{y}|}{|x + \mathbf{t} - y||x + \mathbf{t} - \overline{\underline{y}}|} \zeta_t(y_1, y_2) dy \\ &= \frac{1}{2\pi} \int_{y_1 \geq t, y_2 \geq t} \log \frac{|x + \mathbf{t} - \overline{y}||x + \mathbf{t} - \underline{y}|}{|x + \mathbf{t} - y||x + \mathbf{t} - \overline{\underline{y}}|} \zeta(y_1 - t, y_2 - t) dy \end{aligned}$$

$$= \frac{1}{2\pi}\int_{\Pi_+} \log \frac{|x-\overline{y}+(0,2t)||x-\underline{y}+(2t,0)|}{|x-y||x+y+2\mathbf{t}|}\,\zeta(y)dy. \tag{6.20}$$

Next we define

$$F(s) := \frac{|x-\overline{y}+(0,2s)||x-\underline{y}+(2s,0)|}{|x-y||x+y+2\mathbf{s}|}\,\zeta(y)dy,$$

where $\mathbf{s} = (s,s)$, then it is easy to prove that $F'(s) > 0$ on $[0,\infty)$, hence $F(s) > F(0)$, for all $s > 0$. Recalling (6.20) we infer

$$K_+\zeta_t(x+\mathbf{t}) \geq K_+\zeta(x). \tag{6.21}$$

Now we have

$$\begin{aligned}\int_{\Pi_+} \zeta_t K_+\zeta_t = \int_{x_1\geq t, x_2\geq t} \zeta_t K_+\zeta_t &= \int_{\Pi_+} \zeta_t(x+\mathbf{t})K_+\zeta_t(x+\mathbf{t})dx \\ &= \int_{\Pi_+} \zeta(x_1,x_2)K_+\zeta_t(x+\mathbf{t})dx,\end{aligned}$$

since $\zeta_t(x+\mathbf{t}) = \zeta(x)$, hence by (6.21) we obtain (6.19).$\diamondsuit$

Lemma 11 *Let $2 < p < \infty$ and $\zeta \in L^p(\Pi_+)$ be a non-negative, non-trivial function which vanishes outside $\Pi_+(h)$ for some $h > 0$. Then*

$$K_+\zeta(x) \leq \frac{4hx_1x_2}{\pi|x_1^2 - x_2^2|}\,||\zeta||_1 + N min\{x_1,x_2\}, \tag{6.22}$$

provided $x \in \Pi_+ \setminus diag(\Pi_+)$.

Proof Fix $x \in \Pi_+ \setminus diag(\Pi_+)$ and define

$$U(x) := \left\{ y \in \Pi_+ | \, |(y_1^2 - y_2^2) - (x_1^2 - x_2^2)| < |x_1^2 - x_2^2|^{\frac{1}{2}} \right\}.$$

Next we decompose ζ as follows: $\zeta := \zeta_1 + \zeta_2$, where

$$\zeta_1(y) := \begin{cases} \zeta(y), & y \in \Pi_+(h)\cap U(x) \\ 0, & \text{otherwise.}\end{cases}$$

Again by setting

$$\alpha := |x-\overline{y}|,\ \beta := |x-\underline{y}|,\ \rho := |x-y|,\ \delta := |x-\overline{\underline{y}}|,$$

we obtain

$$
\begin{aligned}
K_+\zeta_2(x) &= \frac{1}{4\pi}\int_{\Pi_+} \log\frac{\alpha^2\beta^2}{\rho^2\delta^2}\,\zeta_2(y)dy = \frac{1}{4\pi}\int_{\Pi_+} \log(1+\frac{16x_1x_2y_1y_2}{\rho^2\delta^2})\,\zeta_2(y)dy \\
&\leq \frac{4hx_1x_2}{\pi}\int_{\Pi_+\setminus U(x)} \frac{1}{\rho^2\delta^2}\,\zeta_2(y)dy. \qquad (6.23)
\end{aligned}
$$

In view of the following identity

$$\rho^2\delta^2 = ((y_1^2-y_2^2)-(x_1^2-x_2^2))^2 + 4(x_1x_2-y_1y_2)^2,$$

we infer that if $y \in \Pi_+ \setminus U(x)$, then $\rho^2\delta^2 > |x_1^2 - x_2^2|$. This in conjunction with (6.23) yields

$$K_+\zeta_2(x) \leq \frac{4hx_1x_2}{\pi|x_1^2-x_2^2|}\,||\zeta||_1. \qquad (6.24)$$

Finally, recalling (6.1) we obtain

$$K_+\zeta_1(x) \leq N\min\{x_1, x_2\}. \qquad (6.25)$$

Since $K_+\zeta(x) = K_+\zeta_1(x) + K_+\zeta_2(x)$, (6.22) follows from (6.24) and (6.25).$\diamondsuit$

Remark Under the hypotheses of Lemma 11 with b replaced by a and an additional assumption, namely, $\Im(\zeta) \geq 1$ we can show existence of a positive constant P such that

$$K_+\zeta(x) \leq P(x_1x_2)^{-\gamma}\,\Im(\zeta), \qquad (6.26)$$

provided $\min\{x_1, x_2\} \geq a/2$ and $\zeta \in \mathcal{F}$. Clearly the truth of (6.26) emerges from the elementary fact that $s^{\gamma-1}\log s$ is bounded on any interval of the form $[d, \infty)$, $d > 0$.

6.4 Proof of the theorem

Proof of Theorem 1. We first show that, for I sufficiently large, there exists a positive constant $R(I)$ such that if $\Pi_+(\xi,\eta)$ is sufficiently large (satisfying $\mathcal{H}(I)$) and $\zeta \in \Sigma(\xi,\eta,I)$, then

$$\text{supp}(\zeta) \subset \Pi_+(R(I)), \qquad (6.27)$$

modulo a set of zero measure. From Lemma 2, there exists $I_3 > I_1$ such that if $I > I_3$, then

$$\sigma(I) > \frac{5}{2}aN||\zeta_0||_1. \qquad (6.28)$$

Fix $I > I_3$ and consider $\zeta \in \Sigma(\xi,\eta,I)$ for some $\Pi_+(\xi,\eta)$ satisfying $\mathcal{H}(I)$. From (6.28) and definition of σ we infer

$$\frac{5}{2}aN\|\zeta\|_1 \leq \Psi(\zeta) \leq \frac{1}{2}\|\zeta\|_1 \sup_{x\in\operatorname{supp}(\zeta)} K_+\zeta(x),$$

thus

$$\sup_{x\in\operatorname{supp}(\zeta)} K_+\zeta(x) \geq 5aN. \tag{6.29}$$

Since $K_+\zeta \in C(\mathbb{R}^2)$ it attains its maximum relative to $\overline{\operatorname{supp}(\zeta)}$ at z, say. Therefore by applying (6.29) we obtain

$$5aN \leq K_+\zeta(z) \leq N\min\{z_1, z_2\},$$

whence $\min\{z_1, z_2\} \geq 5a$. Without loss of generality we may assume that $\Im(\zeta) \geq 1$, hence by (6.26) we obtain

$$5aN \leq K_+\zeta(z) \leq PI(z_1z_2)^{-\gamma},$$

so

$$z_1z_2 \leq \left(\frac{PI}{5aN}\right)^{\gamma}.$$

Let us now define

$$R(I) := \max\left\{\left(\frac{PI}{5aN}\right)^{\gamma},\ 25a^2\right\}. \tag{6.30}$$

Then $V := \{x \in \Pi_+ |\ x_1x_2 \leq R(I),\ \min\{x_1, x_2\} \geq 5a\}$ is not empty and $z \in V$. Note that at least a quadrant of $B_{4a}(x)$, for every $x \in V$, is contained in $\Pi_+(R(I))$ and, in fact, contained in $\Pi_+(\xi_1, R(I))$ for some $\xi_1^2 > R(I)$. By $\Pi_+^t(\xi_1, R(I))$ we denote the translation of $\Pi_+(\xi_1, R(I))$ along $\operatorname{diag}(\Pi_+)$, $\sqrt{2}t$ units. Observe that the family of translations $\{\Pi_+^t(\xi_1, R(I))\}_{0\leq t\leq t_0}$, where $t_0 := (I/\|\zeta_0\|_1)^{1/2}$, is uniformly contained in $\Pi_+(\xi_2, \eta_2)$, for some ξ_2 and η_2 (in fact we can take $\xi_2 = \xi_1 + t_0$). From now on we assume $\xi > \xi_2$ and $\eta > \eta_2$. Since a quadrant of $B_{4a}(z)$, designated by Q, is contained in $\Pi_+(R(I))$ we can apply the Mean Value Inequality and (6.1) to deduce

$$K_+\zeta(x) \geq K_+\zeta(z) - 4aN \geq aN, \quad x \in Q \tag{6.31}$$

where the last inequality is obtained from (6.29). To seek a contradiction let us assume that $E := \operatorname{supp}(\zeta) \setminus \Pi_+(R(I))$ has positive measure and write $\zeta = \zeta_0 + \zeta_1$, where

$$\zeta_1 := \zeta\,\chi_E.$$

Since $|Q| = 4\pi a^2 > |\text{supp}(\zeta)| = \pi a^2$, there exists a measure preserving bijection, denoted T, from E onto a subset of $Q \setminus \text{supp}(\zeta)$, say G, see [41]. Now define

$$\zeta_2 := \zeta_1 \circ T^{-1},$$

on the range of T and zero elsewhere, that is,

$$\zeta_2 = (\zeta_1 \circ T^{-1})\, \chi_{\text{im}(T)},$$

where $\text{im}(T)$ is the range of T, and let $\zeta' := \zeta_0 + \zeta_2$. Clearly $\zeta' \in \mathcal{F}(\xi, \eta)$. Let us show that $\Im(\zeta') < \Im(\zeta)$:

$$\begin{aligned}
\Im(\zeta') &= \int_{\Pi_+} x_1 x_2 \zeta_0 + \int_{\Pi_+} x_1 x_2 \zeta_2 \\
&= \int_{\Pi_+} x_1 x_2 \zeta_0 + \int_{\Pi_+} x_1 x_2\, \zeta_1 \circ T^{-1} \\
&= \int_{\Pi_+} x_1 x_2 \zeta_0 + \int_E (x_1 x_2 \circ T)\, \zeta_1 \\
&< \int_{\Pi_+} x_1 x_2 \zeta_0 + \int_{\Pi_+} x_1 x_2 \zeta_1 = \Im(\zeta).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\Psi(\zeta') - \Psi(\zeta) &= \int_{\Pi_+} (\zeta_2 - \zeta_1) K_+ \zeta + \Psi(\zeta_2 - \zeta_1) \\
&> \int_{\Pi_+} (\zeta_2 - \zeta_1) K_+ \zeta,
\end{aligned}$$

since K_+ is strictly positive, see Chapter 4. Hence

$$\begin{aligned}
\Psi(\zeta') - \Psi(\zeta) &> \int_{\Pi_+} \zeta_2 K_+ \zeta - \int_{\{x \in \Pi_+ | x_1 x_2 > R(I)\}} \zeta_1 K_+ \zeta \\
&\geq aN \int_{\Pi_+} \zeta_2 - \int_{\{x \in \Pi_+ | x_1 x_2 > R(I)\}} \zeta_1 K_+ \zeta,
\end{aligned}$$

by (6.31). Now we proceed to estimate $\int_{\{x \in \Pi_+ | x_1 x_2 > R(I)\}} \zeta_1 K_+ \zeta$. For this purpose we set

$$\text{supp}(\zeta) = J_1 \cup J_2,$$

where

$$J_1 := \left\{ x \in \text{supp}(\zeta) |\ x_1 x_2 > R(I),\ \min\{x_1, x_2\} \geq \frac{a}{2} \right\}$$

and

$$J_2 := \left\{x \in \mathrm{supp}(\zeta) |\ x_1x_2 > R(I),\ \min\{x_1, x_2\} < \frac{a}{2}\right\}.$$

If $x \in J_1$, then by (6.26)

$$K_+\zeta(x) \leq PI(x_1x_2)^{-\gamma} \leq PIR(I)^{-\gamma}.$$

On the other hand if $x \in J_2$, then by (6.1)

$$K_+\zeta(x) \leq N \min\{x_1, x_2\} \leq \frac{a}{2}.$$

Therefore, if $x \in \mathrm{supp}(\zeta_1)$

$$K_+\zeta(x) \leq \max\left\{PIR(I)^{-\gamma}, \frac{aN}{2}\right\}.$$

Let us assume that $R(I)$ is large enough to ensure

$$aN - \max\left\{PIR(I)^{-\gamma}, \frac{aN}{2}\right\} > 0.$$

Therefore we obtain

$$\Psi(\zeta') - \Psi(\zeta) \geq \left(aN - \max\left\{PIR(I)^{-\gamma}, \frac{aN}{2}\right\}\right) \|\zeta_1\|_1 > 0.$$

This implies $\Psi(\zeta') > \Psi(\zeta)$. Finally we define ζ'' to be the function obtained by translating ζ' along $\mathrm{diag}(\Pi_+)$ so that $\Im(\zeta'') = I$. If we denote the amount of translation by t, then it is clear that t is the bigger root of the following algebraic equation

$$\|\zeta'\|_1\, t^2 + 2\left(\int_{\Pi_+} (x_1 + x_2)\zeta'\right) t + \int_{\Pi_+} x_1x_2\zeta' = I. \tag{6.32}$$

Note that t depends on ζ; but we are able to find a uniform bound, independent of ζ, as follows. Solving (6.32) for t yields

$$\begin{aligned} t &= \frac{-\int_{\Pi_+}(x_1 + x_2)\zeta' + ((\int_{\Pi_+}(x_1 + x_2)\zeta')^2 - \|\zeta'\|_1\,(\Im(\zeta') - I))^{\frac{1}{2}}}{\|\zeta'\|_1} \\ &< (\|\zeta'\|_1\,(I - \Im(\zeta')))^{\frac{1}{2}} < \left(\frac{I}{\|\zeta'\|_1}\right)^{\frac{1}{2}}, \end{aligned}$$

as desired. Note that the choices of ξ_2 and η_2 ensure that $\zeta'' \in \mathcal{F}(\xi, \eta, I)$. Now by

Lemma 10 we have

$$\Psi(\zeta'') \geq \Psi(\zeta') > \Psi(\zeta).$$

This is a contradiction to the maximality of ζ. Therefore we have been able to show that if $I > I_3$, then there exists $R(I)$ given by (6.30) such that if $\Pi_+(\xi,\eta)$ is sufficiently large ($\xi \geq \xi_2$ and $\eta \geq \eta_2$) and $\zeta \in \Sigma(\xi,\eta,I)$, then for almost every $x \in \text{supp}(\zeta)$, (6.27) holds.

However, the possibility that the vortex core runs off to infinity, as $\Pi_+(\xi,\eta)$ exhausts Π_+, still exists. We now show that this situation is ruled out once I is sufficiently large. For this purpose let us fix $I > I_3$ and consider $(\zeta,\phi,\lambda) \in M(\xi,\eta,I)$. We claim that if ξ and η are large enough then λ can not be too negative. For this purpose let $\xi \geq \xi_2$ and $\eta \geq \max\{h,\eta_2\}$, ξ_2 and η_2 are as above, where

$$\begin{aligned} h &:= \left(N|\lambda^*|^{-1}+1\right)\ R(I), \\ \lambda^* &:= -\frac{aN}{3R(I)}, \end{aligned}$$

such that $\Pi_+(\xi,\eta)$ satisfies $\mathcal{H}(I)$. We show

$$\lambda > \lambda^*. \tag{6.33}$$

To seek a contradiction suppose $\lambda \leq \lambda^*$. Without loss of generality we may assume that $R(I) \geq 1$. Let $x \in W := \{y \in \Pi_+(\xi,\eta)|\ y_1y_2 > h\}$. Then

$$\begin{aligned} K_+\zeta(x) - \lambda x_1x_2 &> -\lambda x_1x_2 = |\lambda|x_1x_2 > |\lambda|h \\ &= |\lambda|\left(N|\lambda^*|^{-1}+1\right)R(I) > (N+|\lambda|)R(I). \end{aligned}$$

Now consider $x \in \text{supp}(\zeta)$. If $\max\{x_1,x_2\} \geq 1$, then $\min\{x_1,x_2\} \leq x_1x_2$, hence $\min\{x_1,x_2\} \leq R(I)$. If, however, $\max\{x_1,x_2\} < 1$ then $\min\{x_1,x_2\} < 1 \leq R(I)$. Therefore in either case we have $\min\{x_1,x_2\} \leq R(I)$. This, in turn, implies

$$K_+\zeta(x) - \lambda x_1x_2 \leq N\min\{x_1,x_2\} - \lambda x_1x_2 < (N+|\lambda|)R(I),$$

whence

$$\sup_{x\in\text{supp}(\zeta)} (K_+\zeta(x) - \lambda x_1x_2) \leq (N+|\lambda|)R(I).$$

Therefore $K_+\zeta(x) - \lambda x_1x_2$ takes greater values on a non-empty subset of $\Pi_+(\xi,\eta)$, namely W, than its supremum on $\text{supp}(\zeta)$. This is impossible, since ζ is essentially an increasing function of $K_+\zeta(x)-\lambda x_1x_2$ on $\Pi_+(\xi,\eta)$. Hence we derive (6.33). For the rest of the proof we fix $I > I_0 := \max\{I_1,I_2,I_3\}$. Let $\xi > \xi_2$, $\eta > h$ (as above) be such that

$\Pi_+(\xi,\eta)$ satisfies $\mathcal{H}(I)$. Consider $(\zeta,\phi,\lambda) \in M(\xi,\eta,I)$. Now fix $x \in \mathrm{supp}(\zeta) \setminus \mathrm{diag}(\Pi_+)$ such that $\min\{x_1,x_2\} < a/6$. Then by Lemmas 8 and 11, in conjunction with (6.33),

$$\begin{aligned} aN \le K_+\zeta(x) - \lambda x_1 x_2 &\le \frac{4R(I)x_1x_2}{\pi|x_1^2 - x_2^2|}\,\|\zeta\|_1 + N\min\{x_1,x_2\} - \lambda^* x_1x_2 \\ &\le \frac{4R(I)x_1x_2}{\pi|x_1^2 - x_2^2|}\,\|\zeta\|_1 + N\min\{x_1,x_2\} - \lambda^* R(I) \\ &\le \frac{4R(I)x_1x_2}{\pi|x_1^2 - x_2^2|}\,\|\zeta\|_1 + \frac{aN}{6} + \frac{aN}{3}. \end{aligned}$$

Hence

$$|x_1^2 - x_2^2| < \frac{8R(I)\|\zeta_0\|_1}{a\pi N}. \tag{6.34}$$

To summarise, we have shown that if $x \in \mathrm{supp}(\zeta)$ is such that $\min\{x_1,x_2\} > a/6$, then $x \in \Pi_+(R(I)) \cap \{y \in \Pi_+ |\ \min\{y_1,y_2\} > a/6\}$; otherwise x satisfies (6.34). This clearly concludes the existence part of the theorem.

Now consider $\zeta \in \Sigma(I)$. Then there exists $\hat{\xi} > 0$ such that $\overline{\mathrm{supp}(\zeta)}$ is a compact subset of $D(\hat{\xi}) := (0,\hat{\xi}) \times (0,\hat{\xi})$ and according to Lemma 1

$$\zeta = \phi \circ (K_+\zeta - \lambda x_1 x_2), \ \text{ a.e. in } D(\hat{\xi}), \tag{6.35}$$

for some increasing function ϕ and $\lambda \in \mathbb{R}$. Note that from Lemma 8

$$\kappa := \text{ess sup}\{K_+\zeta(x) - \lambda x_1 x_2 |\ x \in \mathrm{supp}(\zeta)\} \ge aN > 0.$$

Since the level sets of $K_+\zeta - \lambda x_1 x_2$, on $\mathrm{supp}(\zeta)$, have zero measure, in particular we have

$$|\{x \in \mathrm{supp}(\zeta) |\ K_+\zeta - \lambda x_1 x_2 = \kappa\}| = 0.$$

Therefore

$$K_+\zeta - \lambda x_1 x_2 > \kappa, \ \text{ a.e. in } \mathrm{supp}(\zeta).$$

Thus we may suppose that $\phi(s) = 0$ for $s \le \kappa$. Now if we define $F(s) := \int_0^s \phi(t)dt$, then Lemma 6 yields

$$2\int_{D(\hat{\xi})} F \circ \psi - 2\lambda I = \int_{\partial D(\hat{\xi})} (F \circ \psi)\ (x \cdot \vec{n}), \tag{6.36}$$

where $\psi := K_+\zeta - \lambda x_1 x_2$. We claim that for $x \in \partial D(\hat{\xi})$ we have $\psi \le \kappa$. Otherwise, by the continuity of ψ we can find $B_\epsilon(x)$ such that $B_\epsilon(x) \cap \mathrm{supp}(\zeta)$ has positive measure, since $\overline{\mathrm{supp}(\zeta)}$ is a compact subset of $D(\hat{\xi})$, and $\psi(s) > \kappa$ for $s \in B_\epsilon(x)$; but this is a contradiction to (6.35). Therefore if $x \in \partial D(\hat{\xi})$ we have $F \circ \psi(x) = 0$. Hence from

(6.36) we deduce $\lambda > 0$, as required.

Now fix $x \in \mathrm{supp}(\zeta)$. Since $\lambda > 0$ we can employ Lemma 8 to obtain

$$aN \leq K_{+}\zeta(x) - \lambda x_1 x_2 < K_{+}\zeta(x) \leq N \min\{x_1, x_2\}.$$

Thus $\min\{x_1, x_2\} \geq a$. This proves the vortex core avoids $\partial\Pi_{+}$. The validity of (6.3) is established as in the theorem in Chapter 5.$\diamondsuit$

Chapter 7

Existence of a steady flow with a bounded vortex past an obstacle

7.1 Introduction

In this chapter we prove existence of a steady planar flow past an obstacle attached to the boundary of a quarter-plane, containing a bounded vortex and approaching an irrotational flow at infinity. To do this we first set up an appropriate variational formulation of the problem and then apply a theory developed by Turkington [44, 45], in conjunction with Burton's theory. Similar work has been done by Badiani [4] in the context of a flow past an obstacle attached to the boundary of a half-plane.

7.2 Notation, definitions and statement of results

We denote by p an arbitrary fixed number in $(2, \infty)$. For any number $r \geq 1$, r^* denotes the conjugate exponent, $1/r + 1/r^* = 1$. For any measurable set $E \subseteq \mathbb{R}^2$ we denote its Lebesgue measure by $|E|$. $B_\xi(x)$ denotes the ball centred at $x \in \mathbb{R}^2$ with radius ξ; in case the centre is the origin we write B_ξ. We say a measurable set E is *dense* at $x \in \mathbb{R}^2$ if the intersection of any ball, centred at x, with E has positive measure. The set of all points, at which E is dense, is denoted by $\mathrm{den}(E)$. The *essential diameter* of a measurable set E, denoted by $\mathrm{diam}(E)$, is defined by

$$\mathrm{diam}(E) := \sup\{|x-y| \mid x, y \in \mathrm{den}(E)\ \}.$$

Let D (the obstacle) be an open, bounded, simply connected set containing the origin in its interior and assume $\overline{D} \subset B_1$. Let $\Omega := \Pi_+ \setminus \overline{D}$ such that $\partial\Omega \in C^2$. For $c > 0$ we define

$$\Omega_c := \{x \in \Pi_+ |\ c^{1/2}x \in \Omega\};$$

and $\Omega_{c,\xi} := \Omega_c \cap B_\xi$.

G, with any subscripts, denotes the Green's function for $-\Delta$ with homogeneous Dirichlet boundary conditions in some domain; clearly when the domain is unbounded the corresponding Green's function is assumed to tend to zero at infinity. In particular, G_+, G, G_1 denote the Green's functions in Π_+, Ω, $\Pi_+ \setminus \overline{B}_1$, respectively. Recall from Chapter 4 that

$$G_+(x,y) = \frac{1}{2\pi} \log \frac{|x-\overline{y}|\ |x-\underline{y}|}{|x-y|\ |x-\underline{\overline{y}}|},\ \ x,y \in \Pi_+,\ x \neq y.$$

Furthermore, it is easy to see that

$$G_1(x,y) := G_+(x,y) - \frac{1}{2\pi} \log \frac{|y||x-\overline{y^*}|\ |y||x-\underline{y^*}|}{|y||x-y^*|\ |y||x-\underline{\overline{y^*}}|},\ x,y \in \Pi_+ \setminus B_1,\ x \neq y.$$

Here "overline", "underline" mean reflection about the x_1-axis, x_2-axis, respectively and "*" indicates inversion with respect to the unit circle.

Remark Let us point out that G_1 can be written simpler but we prefer this representation since it serves more conveniently later.

Of course since Ω may not have a nice geometry we are not able, in general, to write down an explicit formula for G but using the same line of argument as in Chapter 4 one can prove the existence of G. G_c, $c > 0$, denotes the Green's function on Ω_c and we have the following identity

$$G_c(x,y) = G(c^{1/2}x, c^{1/2}y),\ \ x,y \in \Omega_c,\ x \neq y.$$

By applying the Maximum Principle we obtain

$$G_1(x,y) \leq G(x,y) \leq G_+(x,y),$$

where each inequality holds in the positive quadrant.

The integral operator K_+ is defined as in Chapter 5. For a measurable function ζ on Ω and $x \in \mathbb{R}^2$ we define

$$K\zeta(x)\ \ :=\ \ \int_\Omega G(x,y)\,\zeta(y)dy$$

$$K_c\zeta(x) \quad := \quad \int_{\Omega_c} G_c(x,y)\,\zeta(y)dy,$$

whenever the integrals exist.

We define $\eta \in C^2(\Omega) \cap C^1(\overline{\Omega})$ to be a solution of

$$\begin{cases} \Delta\eta &= 0 \text{ in } \Omega \\ \eta &= 0 \text{ on } \partial\Omega \\ \eta &= x_1x_2 + O(|x|^{-2}), \text{ as } |x| \to \infty \\ \nabla\eta &= (x_2, x_1) + O(|x|^{-3}), \text{ as } |x| \to \infty. \end{cases}$$

Existence of η will be addressed in the next section. Next, for a measurable function ζ on Ω we define

$$\begin{aligned} \Psi(\zeta) &:= \frac{1}{2}\int_\Omega \zeta\, K\zeta \\ \Im(\zeta) &:= \int_\Omega \eta\,\zeta, \end{aligned}$$

whenever the integrals exist. Now fix $\lambda > 0$; and let ζ be a measurable function on Ω, then we define

$$\Psi_\lambda(\zeta) := \Psi(\zeta) - \lambda\Im(\zeta).$$

Let us fix $\zeta_0 \in L^p(\Omega)$ which is a non-negative, non-trivial function with compact support and assume $|\text{supp}(\zeta_0)| = \pi a^2$, for some $a > 0$. Moreover, we suppose that $||\zeta_0||_1 = 1$. By $\mathcal{F}$ we denote the set of rearrangements of ζ_0 on Ω which have compact support. We now define the first variational problem

$$P_\lambda : \sup_{\zeta\in\mathcal{F}} \Psi_\lambda(\zeta).$$

The corresponding solution set is denoted by Σ_λ. In order to introduce the second variational problem which is a "rescaled" version of P_λ we need first to define the Turkington-transformations. For this purpose fix $c > 0$ and let ζ be a measurable function on Ω. Then we define

$$\mathcal{C}(\zeta)(x) := c\,\zeta(c^{1/2}x), \;\; x \in \Omega_c. \tag{7.1}$$

The mapping $\mathcal{C}$ as define in (7.1) takes measurable functions on Ω to measurable functions on Ω_c and is referred to the Turkington-transformation. Properties of this mapping will be discussed in the next section. By $\mathcal{F}_c$ we denote the set of all rearrangements of $\mathcal{C}(\zeta_0)$ on Ω_c with compact support. Let us observe that $\mathcal{C} : \mathcal{F} \to \mathcal{F}_c$ is a bijection and in fact the inverse is given by

$$\mathcal{C}^{-1}(\zeta)(x) = c^{-1}\zeta(c^{-1/2}x), \;\; x \in \Omega.$$

It is equally easy to see that

$$\|\mathcal{C}(\zeta)\|_r = c^{1/r^*}\|\zeta\|_r, \;\; |\mathrm{supp}(\mathcal{C}(\zeta))| = 1/c\,|\mathrm{supp}(\zeta)|, \;\; r \geq 1.$$

In particular, $\|\mathcal{C}(\zeta)\|_1 = \|\zeta\|_1$.

For a measurable function ζ on Ω_c, $c > 0$, we define

$$\hat{\Psi}_c(\zeta) := \frac{1}{2}\int_{\Omega_c} \zeta\, K_c\zeta - \int_{\Omega_c} \eta_c\zeta, \tag{7.2}$$

where $\eta_c(x) := c^{-1}\,\eta(c^{1/2}x)$, whenever the integrals exist. Now we define the rescaled variational problem. Fix $c > 0$, then

$$\hat{P}_c : \;\; \sup_{\zeta\in\mathcal{F}_c} \hat{\Psi}_c(\zeta).$$

$\hat{\Sigma}_c$ denotes the corresponding solution set. Further more, for $\xi > 1$ we define

$$\hat{P}_{c,\xi} : \;\; \sup_{\zeta\in\mathcal{F}_{c,\xi}} \hat{\Psi}_c(\zeta),$$

where $\mathcal{F}_{c,\xi}$ is the subset of $\mathcal{F}_c$ comprising functions vanishing outside $\Omega_{c,\xi}$. Let us point out that in order to ensure $\mathcal{F}_{c,\xi} \neq \emptyset$ it is sufficient to impose

$$\xi > \left(\frac{1+4a^2}{c}\right)^{1/2}. \tag{7.3}$$

The solution set for $P_{c,\xi}$ is denoted $\Sigma_{c,\xi}$.

The main results of this chapter are the following theorems.

Theorem 1 *There exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$, P_λ has a solution. If $\hat{\zeta}_\lambda$ is a solution and $\psi_\lambda := K\hat{\zeta}_\lambda$, then ψ_λ satisfies the following semilinear elliptic partial differential equation*

$$-\Delta\psi_\lambda = \phi_\lambda \circ (\psi_\lambda - \lambda\eta), \;\; a.e. \;\, in\, \Omega \tag{7.4}$$

where ϕ_λ is an increasing function, unknown a priori.

Theorem 2 *There exist $c_3 > 0$, $R > 0$ such that if $c \geq c_3$ and $\check{\zeta}_c \in \hat{\Sigma}_c$, then*

$$supp(\check{\zeta}_c) \subset B_R,$$

modulo a set of measure zero.

Theorem 3 *Suppose the sequences $\{c_j\}_{j=1}^{\infty}$ ($\subset \mathbb{R}$) and $\{\check{\zeta}_j\}_{j=1}^{\infty}$ are such that*
(i) $c_3 \leq c_j \to \infty$, as $j \to \infty$.
(ii) $\check{\zeta}_j \in \hat{\Sigma}_{c_j}$, for every j.
Suppose $\hat{x}_j := \int_{\Omega_{c_j}} x\, \check{\zeta}_j$. Then $\hat{x}_j \to x_0$, as $j \to \infty$. Here x_0 is the point where the Routh function (see below) attains its global minimum.

7.3 Outline of the proofs

In this section we breifly describe the main lines of the proofs. First by proving standard results about K, which are inherited by K_c, namely, compactness and strict positivity we make Burton's theory applicable. This means problem $P_{c,\xi}$ is solvable and if $\check{\zeta}_{c,\xi} \in \Sigma_{c,\xi}$, then

$$\text{supp}(\check{\zeta}_{c,\xi}) = \{x \in \Omega_{c,\xi} |\ K_c\check{\zeta}_{c,\xi}(x) - \eta_c(x) \geq \gamma_{c,\xi}\},$$

modulo a set of measure zero. Then it is shown that, by merely making c sufficiently large, $\gamma_{c,\xi}$ can not be too negative. This in conjunction with a neat analysis brings about the estimate

$$\gamma_{c,\xi} \geq \frac{1}{4\pi} \log \frac{c^{1/2}}{2a} + C,$$

for large c, hence

$$\text{supp}(\check{\zeta}_{c,\xi}) \subset \{x \in \Omega_{c,\xi} |\ K_c\check{\zeta}_{c,\xi}(x) - \eta_c(x) > 0\},$$

modulo a set of zero measure. Finally by applying [Chapter 5, Lemma 7] we prove that the support of $\check{\zeta}_{c,\xi}$ is essentially contained in a ball centred at the origin with radius $Ac^{1/2}$, A being a universal constant, for all ξ sufficiently large. Clearly this proves the existence part of Theorem 1. Equation (7.4) is established by applying the usual modification proccess.

Our interest in Theorem 2 was motivated by [19]. In [19] the authors prove existence of a steady 2-dimensional flow in which a finite vortex is in equilibrium with the irrotational flow past an obstacle; even though the domain of the fluid is unbounded the corresponding variational problem is carried over a bounded set S containing a point at which the Routh function attains its minimum on S. Our Theorem 2 confirms this approach by showing that for large c the support of $\check{\zeta}_c$ will be concentrated around the unique global minimiser of the corresponding Routh function. The crucial step is part (i) which is described below.

First we improve the result of Theorem 1 by establishing the "diameter lemma". In this lemma we show that the support of maximisers are uniformly bounded, that is

$$\mathrm{diam}(\mathrm{supp}(\check{\zeta}_c)) < K,$$

for sufficiently large c; here $L(2K)$ is an "L-shape" domain to be specified later. This is then used to prove that for large c

$$\mathrm{supp}(\check{\zeta}_c) \subset L(2K) \cup (\Pi_+ \cap B_{R_1}), \tag{7.5}$$

modulo a set of zero measure, for some constant R_1, independent of c. Next it is shown that (7.5) can be improved; in fact we prove the following

$$\mathrm{supp}(\check{\zeta}_c) \subset \{x \in \Pi_+ |\ x_1 x_2 < \alpha\},$$

modulo a set of zero measure, where α is a positive constant independent of c. Therefore the only possibility, for Theorem 2(i) not to be true, is that $\mathrm{supp}(\check{\zeta}_c)$ runs off to infinity by squeezing between the curve $x_1 x_2 = \alpha$ and $\partial\Pi_+$. This will be ruled out in the last step of the proof.

7.4 Preliminary results

We begin with a discussion about the Green's functions G_+, G and G_1; in particular we write each as the difference of the singular and the harmonic parts. This is then followed by some estimates concerning the harmonic functions. We then move on to discussing properties of K. The common scenario is to show that $K : L^p(U) \to L^{p^*}(U)$ is compact and strictly positive so that Burton's theory can be applied to the variational problem $P_{c,\xi}$. To prove strict positivity we first derive some asymptotic estimates.

The next topic will be the existence of η. This is done using the same ideas as in Chapter 3. The rest of this section is devoted to a series of lemmas which will be used in the proof of Theorems 1 and 2.

7.4.1 The Green's functions

The existence of G is proved exactly as in [Chapter 3, section 3.3.1]. By applying the Maximum Principle we obtain

$$G_1(x,y) \le G(x,y) \le G_+(x,y), \tag{7.6}$$

where each inequality is valid in the positive domain. Let us now recall that

$$\begin{aligned}
G(x,y) &= \frac{1}{2\pi}\log\frac{1}{|x-y|} - h(x,y) \\
G_+(x,y) &= \frac{1}{2\pi}\log\frac{1}{|x-y|} - h_+(x,y) \\
G_1(x,y) &= \frac{1}{2\pi}\log\frac{1}{|x-y|} - h_1(x,y),
\end{aligned}$$

where h, h_+ and h_1 are harmonic functions, say for fixed y, in their respective domains. More specifically we have

$$\begin{aligned}
h_+(x,y) &= \frac{1}{2\pi}\log\frac{|x-\underline{\overline{y}}|}{|x-\overline{y}||x-\underline{y}|} \\
h_1(x,y) &= h_+(x,y) + \frac{1}{2\pi}\log\frac{|y||x-\overline{y^*}||y||x-\underline{y^*}|}{|y||x-y^*||y||x-\underline{\overline{y^*}}|}.
\end{aligned}$$

From (7.6) we obtain the following

$$h_+(x,y) \le h(x,y) \le h_1(x,y), \tag{7.7}$$

where the inequalities are understood to hold in the positive domains. Next we set $\hat{h} = h - h_+$ and $\hat{h}_1 = h_1 - h_+$. Then from (7.7) we infer $0 \le \hat{h} \le \hat{h}_1$. Let us now point out that

$$\hat{h}_1(x,y) = \frac{1}{2\pi}\log\frac{|y||x-\overline{y^*}|}{|y||x-y^*|} + \frac{1}{2\pi}\log\frac{|y||x-\underline{y^*}|}{|y||x-\underline{\overline{y^*}}|}.$$

Therefore if we set $\beta_1 := |y|^2|x-\underline{y^*}|^2$ and $\beta_2 := |y|^2|x-\underline{\overline{y^*}}|^2$, then $\beta_1 = \beta_2 - 4x_2y_2$, hence $\log\beta_1/\beta_2 < 0$. This implies

$$\begin{aligned}
0 \le \hat{h}_1(x,y) \le \frac{1}{2\pi}\log\frac{|y||x-\overline{y^*}|}{|y||x-y^*|} &= \frac{1}{4\pi}\log\left(1+\frac{4x_2y_2}{|y|^2|x-y^*|^2}\right) \\
&\le \frac{x_2y_2}{\pi(|x||y|-1)^2},
\end{aligned} \tag{7.8}$$

provided $x, y \in \Pi_+ \setminus \overline{B}_1$. Similarly, we obtain

$$0 \le \hat{h}_1(x,y) \le \frac{x_1 y_1}{\pi(|x||y|-1)^2}, \tag{7.9}$$

provided $x, y \in \Pi_+ \setminus \overline{B}_1$.

7.4.2 Properties of the operator K

Let $\zeta \in L^p(\Omega)$ have compact support. Then $K\zeta$ is defined at every point $x \in \mathbb{R}^2$. This follows immediately from the fact that for $x \in \mathbb{R}^2$

$$|K\zeta(x)| \le K|\zeta|(x) \le K_+|\zeta|(x).$$

Our first lemma can be proved using the same method applied to [Chapter 3, Lemma 3].

Lemma 1 *Let $q \in [1,\infty)$ and let U be an open, bounded subset of Ω. Then $K : L^p(U) \to L^q(U)$ is compact. Moreover, if $\zeta \in L^p(\Omega)$ vanishes outside U, then*
(i) $-\Delta K\zeta = \zeta$, a.e. in Ω
(ii) $K\zeta = 0$, on $\partial\Omega$
(iii) $K\zeta \in W^{2,p}_{loc}(\overline{\Omega})$, that is, for every open and bounded set $O \subset \Omega$ with $\overline{O} \subset \overline{\Omega}$ we have $K\zeta \in W^{2,p}(O)$.

The next lemma is used to prove strict positivity of K.

Lemma 2 *Let $\zeta \in L^p(\Omega)$ have compact support. Then $K\zeta(x) = O(|x|^{-1})$, $\nabla K\zeta(x) = O(|x|^{-2})$, as $|x| \to \infty$.*

Proof Let us recall that

$$|K\zeta(x)| \le K_+|\zeta|(x),$$

for every $x \in \mathbb{R}^2$. Since $K_+|\zeta|(x) = O(|x|^{-1})$ as $|x| \to \infty$, see Chapter 4, we deduce that $K\zeta(x) = O(|x|^{-1})$ as $|x| \to \infty$. Hence if $A > 0$, then there exists $M_1 > 0$ such that for $|x| > M_1$ we have $|K\zeta(x)| \le A|x|^{-1}$. Now let us consider a special extension of $K\zeta$, denoted $(K\zeta)_e$, which is defined by

$$(K\zeta)_e = \begin{cases} K\zeta(x) & x \in \Omega \\ -K\zeta(\underline{x}) & x \in \Omega_- \\ -K\zeta(\overline{x}) & x \in \Omega^-, \end{cases}$$

where Ω_-, Ω^- denote the reflection of Ω about the lines $x_1 = 0$, $x_2 = 0$, respectively. Let us note that $(K\zeta)_e$ is harmonic in $\overline{\Omega} \cup \Omega^- \cup \Omega_- \setminus B_{M_2}$, for some $M_2 > 0$. Now consider x such that $|x| > M := \max\{2, 2M_1, 2M_2\}$, then by Harnack's inequality [28,Theorem 2.10, p23] we obtain

$$|\nabla(K\zeta)_e(x)| \le \frac{8}{|x|} \sup_{z \in B_{|x|/2}(x)} |(K\zeta)_e(z)| \le 8A|x|^{-2}.$$

Hence we are done.$\diamondsuit$

The next lemma can be proved using the same method as in [Chapter 4, Lemma 4] along with Lemma 2.

Lemma 3 *Let q and U be as in Lemma 1. Then $K : L^p(U) \to L^q(U)$ is strictly positive, that is, for every non-trivial $\zeta \in L^p(\Omega)$, vanishing outside U,*

$$\int_\Omega \zeta \, K\zeta > 0.$$

Clearly all of the properties of K encapsulated in Lemmas 1, 2 and 3 are inherited by K_c, $c > 0$. In particular if U is an open, bounded subset of $\Omega_{c,\xi}$, where ξ satisfies (7.3), then $K_c : L^p(U) \to L^{p^*}(U)$ is compact and strictly positive, hence Burton's theory can be applied to deduce that $P_{c,\xi}$ has a solution.

7.4.3 Existence of η

In this section we prove

Lemma 4 *There exists $\eta \in C^2(\Omega) \cap C^1(\overline{\Omega})$ which satisfies*

$$\begin{cases} \Delta\eta &= 0 \quad in\, \Omega \\ \eta &= 0 \quad on\, \partial\Omega \\ \eta &= x_1x_2 + O(|x|^{-2}), \quad as\, |x| \to \infty \\ \nabla\eta &= (x_2, x_1) + O(|x|^{-3}), \quad as\, |x| \to \infty. \end{cases}$$

Proof Define $\Omega_n := \Omega \cap B_n$ for $n \in \mathbb{N}$. Let η_n denote the classical solution of the following boundary value problem

$$\begin{cases} \Delta\eta_n = 0 & \text{in} \quad \Omega_n \\ \eta_n = 0 & \text{on} \quad \partial\Omega_n \cap \partial\Omega \\ \eta_n = x_1x_2 & \text{on} \quad \partial\Omega_n \setminus \partial\Omega. \end{cases}$$

Therefore $\eta_n \in C^2(\Omega_n) \cap C(\overline{\Omega}_n)$, by [28, Chapter 2, Theorem 2.14]. Now by an application of the Maximum Principle we deduce, for every $x \in \Omega_n$

$$0 \le \eta_n(x) \le x_1x_2, \quad \eta_{n+1}(x) \le \eta_n(x).$$

Whence for every $x \in \overline{\Omega}$ the following limit exists

$$\eta(x) := \lim_{n\to\infty} \eta_n(x). \tag{7.10}$$

By [22, Theorem 8, p151], η is harmonic in Ω and for every compact subset of Ω the convergence in (7.10) is uniform. We now proceed to show boundary regularity of η. We first prove that $\eta \in C(\overline{\Omega})$. Let us consider $x_0 \in \partial\Omega$ and $n \in \mathbb{N}$ such that $x_0 \in \partial\Omega_n$. Since

$$0 \leq \eta(x) \leq \eta_n(x),$$

for every $x \in \overline{\Omega}_n$, and $\eta(x_0) = 0$, it follows that

$$|\eta(x) - \eta(x_0)| \leq |\eta_n(x) - \eta_n(x_0)|,$$

for every $x \in \overline{\Omega}_n$. This, in turn, implies that η is continuous at x_0, since η_n is continuous at x_0.

To show that indeed $\eta \in C^1(\overline{\Omega})$ we fix $n_0 \in \mathbb{N}$. Observe that $\Gamma_2 := \partial\Omega_{n_0} \setminus \overline{\partial B_{n_0}}$ is a relatively open subset of $\partial\Omega_{n_0}$. Moreover, since $\partial\Omega \in C^2$, Γ_2 is a regular manifold of class $W^{2,\infty}$, see [17, vol 1, p146]. Thus we can apply [17, Proposition 1(2), p146] to deduce $\eta \in C^1(\Omega_{n_0} \cap \Omega_2)$. Since n_0 is arbitrary we obtain the desired result.

Finally we establish the asymptotic estimates. First observe that $x_1x_2 - x_1x_2/|x|^4$ is harmonic in the punctured plane, $\mathbb{R}^2 \setminus (0,0)$, being the difference of two harmonic functions. Now by applying the Maximum Principle we obtain

$$x_1x_2 - \frac{x_1x_2}{|x|^4} \leq \eta_n(x) \leq x_1x_2, \quad x \in \Omega_n$$

for every $n \in \mathbb{N}$, thanks to our assumption $\overline{D} \subset B_1$. Therefore

$$x_1x_2 - \frac{x_1x_2}{|x|^4} \leq \eta(x) \leq x_1x_2, \quad x \in \Omega. \tag{7.11}$$

Now define η^* as follows

$$\eta^*(x) := \begin{cases} \eta(x) & x \in \overline{\Omega} \\ -\eta(\underline{x}) & x \in \Omega_- \\ -\eta(\overline{x}) & x \in \Omega^-. \end{cases}$$

Hence

$$x_1x_2 \leq \eta^*(x) \leq x_1x_2 - \frac{x_1x_2}{|x|^4},$$

for all $x \in \Omega^- \cup \Omega_-$. Therefore if $u := \eta^* - x_1 x_2$, then

$$0 \le |u(x)| \le \frac{1}{|x|^2}, \quad x \in \Omega.$$

This implies $\eta(x) = x_1 x_2 + O(|x|^{-2})$ as $|x| \to \infty$. Now consider x such that $|x|/2 > 1$. Then by Harnack's inequality we have

$$|\nabla u(x)| \le \frac{4}{|x|} \sup_{z \in \partial B_{|x|/2}(x)} |u(z)| \le \sup_{z \in \partial B_{|x|/2}(x)} \frac{4}{|x||z|^2} \le \frac{16}{|x|^3},$$

since for $z \in \partial B_{|x|/2}(x)$ we have $|x| - |z| \le |x - z| = |x|/2$, hence $1/|z|^2 < 4/|x|^2$. So we are done.$\diamondsuit$

7.4.4 Properties of Turkington-transformations

Let $c > 0$ and define $\mathcal{C}$ as in (7.1). In this section we prove

Lemma 5 *$\mathcal{C} : \mathcal{F} \to \mathcal{F}_c$ is a bijection. Moreover, if $\zeta \in \mathcal{F}$, then*
(i) $\|\mathcal{C}(\zeta)\|_p = c^{1/p^*} \, \|\zeta\|_p$
(ii) $|supp(\mathcal{C}(\zeta))| = c^{-1} \, |supp(\zeta)|$.

Proof Let us first prove that the mapping $\mathcal{C} : \mathcal{F} \to \mathcal{F}_c$ is well defined. Consider $\zeta \in \mathcal{F}$ and define

$$\lambda_\zeta(s) := |\{x \in \Omega | \; \zeta(x) \ge s\}|, \;\; s > 0.$$

$\lambda_\zeta(\cdot)$ is called the distribution function of ζ (see Remark 1 below). Next we define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $Tx = c^{1/2}x$. Let us now fix $s > 0$ and calculate

$$\lambda_{\mathcal{C}(\zeta)}(s) = \int_{\Omega_c} \chi_{T^{-1} \circ \zeta^{-1}([s/c,\infty))} = \int_{\Omega_c} \chi_{\zeta^{-1}([s/c,\infty))} \circ T.$$

Here χ_E is the characteristic function of E and the exponent "-1" stands for the "pre-image". Using the change of variable formula [30] we obtain

$$\lambda_{\mathcal{C}(\zeta)}(s) = c^{-1} \int_{\Omega} \chi_{\zeta^{-1}([s/c,\infty))} = c^{-1} \, \lambda_\zeta(s/c). \tag{7.12}$$

Since ζ is a rearrangement of ζ_0 we have $\lambda_\zeta(s/c) = \lambda_{\zeta_0}(s/c)$. Hence $\lambda_{\mathcal{C}(\zeta)}(s) = c^{-1}\lambda_{\zeta_0}(s/c)$. From (7.12) we deduce $c^{-1}\lambda_{\zeta_0}(s/c) = \lambda_{\mathcal{C}(\zeta_0)}(s)$, hence $\lambda_{\mathcal{C}(\zeta)}(s) = \lambda_{\mathcal{C}(\zeta_0)}(s)$. Therefore $\mathcal{C}(\zeta) \in \mathcal{F}_c$.

Proof of the bijectivity of $\mathcal{C}$ is trivial; the inverse of $\mathcal{C}$ is, for $\check{\zeta} \in \mathcal{F}_c$,

$$\mathcal{C}^{-1}(\check{\zeta})(x) = c^{-1} \, \check{\zeta}(c^{-1/2}x),$$

for every $x \in \Omega$. (i) is derived by straightforward calculations. Let us prove (ii). We clearly have

$$|\text{supp}(\mathcal{C}(\zeta))| = |\{x \in \Omega_c | \ \zeta(c^{1/2}x) > 0\}| = \int_{\Omega_c} \chi_{\zeta^{-1}((0,\infty))} \circ T.$$

Again by changing variables we obtain

$$\int_{\Omega_c} \chi_{\zeta^{-1}((0,\infty))} \circ T = c^{-1} \int_{\Omega_c} \chi_{\zeta^{-1}((0,\infty))} = c^{-1}|\text{supp}(\zeta)|.$$

Hence we are done.◇

Remark 1 Sometimes in the literature the distribution function is defined with strict inequality; in this case the function turns out to be right continuous whereas in our situation it is left continuous.

Remark 2 Let us point out that according to Lemma 5 it is readily verified that if $\check{\zeta} \in \Sigma_c$, then $\mathcal{C}^{-1}(\check{\zeta}) \in \Sigma_\lambda$, for $\lambda = 1/c$.

7.4.5 The Routh function

In our analysis we make use of the so called " Routh function " which is defined by $H(x) = H_1(x) + H_2(x)$, $x \in \Pi_+$, where

$$\begin{aligned} H_1(x) &= \frac{1}{4\pi} \log \frac{|x|}{2x_1x_2}, \\ H_2(x) &= x_1x_2. \end{aligned}$$

Observe that for $z \in \partial\Pi_+$ we have $\lim_{x\to z} H(x) = \infty$. Elementary calculations prove that H has a unique global minimum at $x_0 = (1/(2\sqrt{2\pi}), (1/(2\sqrt{2\pi}))$. Occasionally the first and the second co-ordinates of x_0 are denoted by $x_{0,1}$ and $x_{0,2}$.

For more information on Routh function the reader is referred to the classic monograph [35].

7.4.6 Some more auxiliary lemmas

This section is devoted to some crucial lemmas which will be used to prove Theorems 1 and 2.

Lemma 6 *Let c and ξ be positive constants satisfying (7.1). Let $\check{\zeta}_{c,\xi} \in \Sigma_{c,\xi}$; then*

$$\hat{\Psi}_c(\check{\zeta}_{c,\xi}) \geq \frac{1}{4\pi} \log \frac{c^{1/2}}{2a} - C_1, \tag{7.13}$$

where C_1 is a positive constant, provided c is sufficiently large.

Proof Let $\zeta^*_{c,\xi}$ denote the Schwarz-symmetrisation of $\check{\zeta}_{c,\xi}$ with respect to x_0 (see section 6.3.5). By Lemma 5(ii) we have $\mathrm{supp}(\zeta^*_{c,\xi}) = B_{a/c^{1/2}}(x_0)$, hence, for sufficiently large c, we can ensure $\mathrm{supp}(\zeta^*_{c,\xi}) \subset \Omega_{c,\xi}$. Thus $\hat{\Psi}_c(\check{\zeta}_{c,\xi}) \geq \hat{\Psi}_c(\zeta^*_{c,\xi})$. We now proceed to estimate $\hat{\Psi}_c(\zeta^*_{c,\xi})$ from below. From the definition of $\hat{\Psi}_c$ we have

$$\begin{aligned}
\hat{\Psi}_c(\zeta^*_{c,\xi}) &= \frac{1}{2}\int_{\Omega_c} \zeta^*_{c,\xi}\, K_c \zeta^*_{c,\xi} - \int_{\Omega_c} \eta_c\, \zeta^*_{c,\xi} \\
&= \int_{\Omega_c}\int_{\Omega_c} \left(\frac{1}{4\pi} \log \frac{1}{c^{1/2}|x-y|} - \frac{1}{2} h(c^{1/2}x, c^{1/2}y) \right) \zeta^*_{c,\xi}(x) \zeta^*_{c,\xi}(y) dx dy \\
&\quad - \int_{\Omega_c} \eta_c(x)\, \zeta^*_{c,\xi}(x) dx = I_1 - I_2.
\end{aligned}$$

We now estimate I_1 as follows

$$\begin{aligned}
I_1 &\geq \frac{1}{4\pi} \log \frac{1}{2a} \\
&\quad - \int_{\Omega_c}\int_{\Omega_c} \left(\frac{1}{2} h(c^{1/2}x, c^{1/2}y) - H_1(c^{1/2}x_0) \right) \zeta^*_{c,\xi}(x) \zeta^*_{c,\xi}(y) dx dy \\
&\quad - H_1(c^{1/2}x_0),
\end{aligned} \tag{7.14}$$

where we have used $\|\zeta^*_{c,\xi}\|_1 = 1$. Now we show the integral in (7.14), denoted by J_1, is $o(1)$ as $c \to \infty$. First note that for $x,\ y \in \mathrm{supp}(\zeta^*_{c,\xi})$ we have

$$\begin{aligned}
\left| \frac{1}{2} h(c^{1/2}x, c^{1/2}y) - H_1(c^{1/2}x_0) \right| &\leq \frac{1}{2} \hat{h}_1(c^{1/2}x, c^{1/2}y) \\
&\quad + \left| \frac{1}{2} h_+(c^{1/2}x, c^{1/2}y) - H_1(c^{1/2}x_0) \right|.
\end{aligned}$$

Now by applying (7.8) we deduce

$$\begin{aligned}
\left| \frac{1}{2} h(c^{1/2}x, c^{1/2}y) - H_1(c^{1/2}x_0) \right| &\leq \frac{c x_2 y_2}{2\pi(c|x||y|-1)^2} \\
&\quad + \left| \frac{1}{2} h_+(c^{1/2}x, c^{1/2}y) - H_1(c^{1/2}x_0) \right|,
\end{aligned}$$

for sufficiently large c. Next observe that

$$\sup_{x,y\in \operatorname{supp}(\zeta^*_{c,\xi})} \frac{cx_2y_2}{2\pi(c|x||y|-1)^2} \to 0,$$

as $c \to \infty$; also

$$\begin{aligned} &\sup_{x,y\in \operatorname{supp}(\zeta^*_{c,\xi})} \left|\frac{1}{2}h_0(c^{1/2}x, c^{1/2}y) - H_1(c^{1/2}x_0)\right| \\ = &\sup_{x,y\in \operatorname{supp}(\zeta^*_{c,\xi})} \left|\frac{1}{4\pi}\log\frac{2|x-\overline{y}|x_{0,1}x_{0,2}}{|x-\overline{y}||x-\underline{y}||x_0|}\right| \to 0, \end{aligned}$$

as $c \to \infty$, where

$$h_0(x,y) = \frac{1}{2\pi}\log\frac{|x-\overline{y}|}{|x-\overline{y}||x-\underline{y}|}.$$

Therefore $J_1 = o(1)$ as $c \to \infty$, whence

$$I_1 \geq \frac{1}{4\pi}\log\frac{1}{2a} - H_1(c^{1/2}x_0) - o(1), \tag{7.15}$$

as $c \to \infty$.

Now we estimate I_2,

$$\begin{aligned} I_2 &= \int_{\Omega_c} \eta_c(x)\, \zeta^*_{c,\xi}(x)dx \\ &= \int_{\Omega_c} (\eta_c(x) - x_{0,1}x_{0,2})\, \zeta^*_{c,\xi}(x)dx + H_2(x_0). \end{aligned}$$

Note that from (7.11) we infer

$$x_1x_2 - \frac{x_1x_2}{c^2|x|^4} \leq \eta_c(x) \leq x_1x_2, \;\; x \in \Omega_c.$$

Hence

$$\begin{aligned} &\sup_{x,y\in \operatorname{supp}(\zeta^*_{c,\xi})} |\eta_c(x) - x_{0,1}x_{0,2}| \leq \\ &\sup_{x,y\in \operatorname{supp}(\zeta^*_{c,\xi})} (2|x_1x_2 - x_{0,1}x_{0,2}| + 1/(c^2|x|^2)) \to 0, \end{aligned}$$

as $c \to \infty$. Therefore

$$I_2 = H_2(x_0) + o(1), \tag{7.16}$$

as $c \to \infty$. Now from (7.15) and (7.16) we deduce

$$\hat{\Psi}_c(\zeta^*_{c,\xi}) \geq \frac{1}{4\pi} \log \frac{1}{2a} - H_1(c^{1/2}x_0) - H_2(x_0) + o(1),$$

as $c \to \infty$, or

$$\hat{\Psi}_c(\zeta^*_{c,\xi}) \geq \frac{1}{4\pi} \log \frac{c^{1/2}}{2a} - H(x_0) + o(1),$$

as $c \to \infty$. This clearly verifies (7.13).◇

According to the last paragraph in section (6.3.2), if c and ξ satisfy (7.3), then $P_{c,\xi}$ has a solution. Whence, by Burton's theory, see [Chapter 4, section 4.3.2], if $\check{\zeta}_{c,\xi} \in \Sigma_{c,\xi}$, then there exists an increasing function $\phi_{c.\xi}$ such that

$$\check{\zeta}_{c,\xi} = \phi_{c,\xi} \circ (K_c\check{\zeta}_{c,\xi} - \eta_c) \text{ a.e. in } \Omega_{c,\xi}.$$

From this it follows

$$\operatorname{supp}(\check{\zeta}_{c,\xi}) = \{x \in \Omega_{c,\xi} | \; K_c\check{\zeta}_{c,\xi}(x) - \eta_c(x) \geq \gamma_{c,\xi}\}, \tag{7.17}$$

for some constant $\gamma_{c,\xi}$, modulo a set of measure zero. Note that the inequality in (7.17) can be changed to strict inequality, since the level sets of $K_c\check{\zeta}_{c,\xi} - \eta_c$ (sets on which $K_c\check{\zeta}_{c,\xi} - \eta_c$ is constant) on $\operatorname{supp}(\check{\zeta}_{c,\xi})$ have measure zero, by [28, Chapter 7, Lemma 7.7]. In the next lemma we derive a lower bound for $\gamma_{c,\xi}$ when c and ξ are sufficiently large.

Lemma 7 *There exists $c_1 > 0$ and $\xi_1 > 0$ such that if $c \geq c_1$ and $\xi \geq \xi_1$, then*

$$\gamma_{c,\xi} \geq \frac{1}{2\pi} \log \frac{c^{1/2}}{2a} + C_2(k), \tag{7.18}$$

where $C_2(k)$ is a constant depending on k, the constant depending on the cone determining the cone property of $\Omega_{c,\xi}$.

Proof Let c_1' and ξ_1' be positive constants such that if $c \geq c_1'$ and $\xi \geq \xi_1'$, satisfy (7.3), then $B := B_{1/(2\sqrt{2\pi})}(x_0) \subset \Omega_{c,\xi}$. Let $\tilde{\gamma} > 0$ be such that $B \subset \Pi_+(\tilde{\gamma}) := \{x \in \Pi_+ | \; x_1x_2 < \tilde{\gamma}\}$. We claim that by merely enlarging c we can ensure $\gamma_{c,\xi} \geq -\tilde{\gamma}$. To seek a contradiction suppose $\gamma_{c,\xi} < -\tilde{\gamma}$; then for $x \in B$ we have

$$K_c\check{\zeta}_{c,\xi}(x) - \eta_c(x) \geq -\eta_c(x) \geq -x_1x_2 > -\tilde{\gamma} > \gamma_{c,\xi},$$

since $K_c\check{\zeta}_{c,\xi}$ is non-negative and $\eta_c \leq x_1x_2$. Therefore $B \subset \operatorname{supp}(\check{\zeta}_{c,\xi})$, modulo a set of measure zero. Hence $|B| \leq |\operatorname{supp}(\check{\zeta}_{c,\xi})|$, that is, $1/8 \leq \pi a^2/c$, so $c \leq 8\pi a^2$. To derive a contradiction it suffices to make c greater than $8\pi a^2$. Henceforth we assume $c \geq \max\{c_1', 8\pi a^2\}$. Therefore we obtain

$$\operatorname{supp}(\check{\zeta}_{c,\xi}) \subseteq \{x \in \Omega_{c,\xi} |\ K_c\check{\zeta}_{c,\xi}(x) - \eta_c(x) > -\tilde{\gamma}\}, \tag{7.19}$$

modulo a set of measure zero. Let us now define the " adjusted " energy functional

$$F(\zeta) := \int_{\Omega_c} (K_c\zeta - \eta_c - \gamma_{c,\xi})\, \zeta,$$

for measurable functions ζ on Ω_c. Observe that

$$F(\check{\zeta}_{c,\xi}) \leq \frac{1}{2}\int_{\Omega_c} u^+\, \check{\zeta}_{c,\xi} - \frac{1}{2}(\gamma_0 - 1),$$

where $\gamma_0 := -\tilde{\gamma}$ and $u := K_c\check{\zeta}_{c,\xi} - \eta_c - \gamma_{c,\xi} + \gamma_0 - 1$. It is clear that there exists $M > \xi$ such that $u^+ \in H_0^1(\Omega_{c,M})$. Now by applying the " half Green formula ", see [29, formula (1.5.1), p24], and Lemma 1(ii), applied to $K_c\check{\zeta}_{c,\xi}$ we find

$$\begin{aligned} \|\nabla u^+\|^2_{2,\Omega_{c,\xi}} &\leq \|\nabla u^+\|^2_{2,\Omega_{c,M}} = \int_{\Omega_{c,M}} \nabla u^+ \cdot \nabla u^+ \\ &= \int_{\Omega_{c,M}} \nabla u^+ \cdot \nabla u = -\int_{\Omega_{c,\xi}} u^+\, \check{\zeta}_{c,\xi}. \end{aligned}$$

Hence we can apply Hölder's inequality to obtain

$$\|\nabla u^+\|^2_{2,\Omega_{c,\xi}} \leq \|u^+\|_{2,\Omega_{c,\xi}}\, \|\check{\zeta}_{c,\xi}\|_{2,\Omega_{c,\xi}}, \tag{7.20}$$

where we have used Lemma 5(i). From the continuous embedding

$$W^{1,1}(\Omega_{c,\xi}) \hookrightarrow L^2(\Omega_{c,\xi}),$$

see [1, p105], we deduce

$$\|u^+\|_{2,\Omega_{c,\xi}} \leq k\, \|u^+\|_{1,1,\Omega_{c,\xi}},$$

where k is the constant depending on the cone determining the cone property of $\Omega_{c,\xi}$; let us point out that the cone is independent of c and ξ, hence, in turn, k is independent of c and ξ. Next we observe

$$\|u^+\|_{1,1,\Omega_{c,\xi}} \leq \|u^+\|_{2,\Omega_{c,\xi}} + 2\|\nabla u^+\|_{1,\Omega_{c,\xi}}.$$

Note that supp(u^+) and supp(∇u^+) are both contained in supp(u); and since supp(u) is essentially contained in supp$(\check{\zeta}_{c,\xi})$ we deduce that supp(u^+) and supp(∇u^+) are essentially contained in supp$(\check{\zeta}_{c,\xi})$. This implies

$$\|u^+\|_{1,\Omega_{c,\xi}} = \|u^+\|_{1,\text{supp}(\check{\zeta}_{c,\xi})} \leq |\text{supp}(\check{\zeta}_{c,\xi})|^{1/2} \, \|u^+\|_{2,\Omega_{c,\xi}} = \frac{\sqrt{\pi}a}{c^{1/2}} \, \|u^+\|_{2,\Omega_{c,\xi}},$$

where we have used Hölder's inequality and Lemma 5(ii). Similarly we obtain

$$\|\nabla u^+\|_{1,\Omega_{c,\xi}} \leq \frac{\sqrt{\pi}a}{c^{1/2}} \, \|\nabla u^+\|_{2,\Omega_{c,\xi}}.$$

Therefore we derive

$$\|u^+\|_{2,\Omega_{c,\xi}} \leq k\left(\frac{\sqrt{\pi}a}{c^{1/2}} \, \|u^+\|_{2,\Omega_{c,\xi}} + \frac{2\sqrt{\pi}a}{c^{1/2}} \, \|\nabla u^+\|_{2,\Omega_{c,\xi}}\right).$$

This, in turn, implies

$$\left(1 - \frac{k\sqrt{\pi}a}{c^{1/2}}\right) \, \|u^+\|_{2,\Omega_{c,\xi}} \leq \frac{2k\sqrt{\pi}a}{c^{1/2}} \, \|\nabla u^+\|_{2,\Omega_{c,\xi}}.$$

Let c_1'' be a positive number such that $c \geq c_1''$ implies $(1 - k\sqrt{\pi}a/c^{1/2}) > 1/2$, then for $c \geq \max\{c_1', c_1'', 8\pi a^2\}$ we obtain

$$\frac{1}{2}\|u^+\|_{2,\Omega_{c,\xi}} \leq \frac{2k\sqrt{\pi}a}{c^{1/2}} \, \|\nabla u^+\|_{2,\Omega_{c,\xi}}. \tag{7.21}$$

From (7.20) and (7.21) we deduce

$$\|\nabla u^+\|^2_{2,\Omega_{c,\xi}} \leq 4k\sqrt{\pi}a\|\zeta_0\|_2 \, \|\nabla u^+\|_{2,\Omega_{c,\xi}},$$

hence

$$\|\nabla u^+\|_{2,\Omega_{c,\xi}} \leq 4k\sqrt{\pi}a\|\zeta_0\|_2. \tag{7.22}$$

Therefore by applying Hölder's inequality, (7.21) and (7.22) we obtain

$$\begin{aligned}\int_{\Omega_{c,\xi}} u^+ \, \check{\zeta}_{c,\xi} \leq c^{1/2}\|\zeta_0\|_2 \, \|u^+\|_{2,\Omega_{c,\xi}} &\leq 4k\sqrt{\pi}a\|\zeta_0\|_2 \, \|\nabla u^+\|_{2,\Omega_{c,\xi}} \\ &\leq (4k\sqrt{\pi}a\|\zeta_0\|_2)^2 =: \tilde{\beta}(k).\end{aligned}$$

From this we infer

$$F(\check{\zeta}_{c,\xi}) \leq \frac{1}{2}(\tilde{\beta}(k) - \gamma_0 + 1) =: \beta(k).$$

Therefore we derive

$$\hat{\Psi}_c(\check{\zeta}_{c,\xi}) = F(\check{\zeta}_{c,\xi}) - \frac{1}{2}\int_{\Omega_{c,\xi}} \eta_c \, \check{\zeta}_{c,\xi} + \frac{1}{2}\gamma_{c,\xi} \leq F(\check{\zeta}_{c,\xi}) + \frac{1}{2}\gamma_{c,\xi}. \tag{7.23}$$

By Lemma 6 there exists $c_1''' > 0$ such that for $c \geq c_1'''$ we have $\hat{\Psi}_c(\check{\zeta}_{c,\xi}) \geq 1/(4\pi)\log c^{1/2}/(2a) - C_1$. Hence if $c \geq \max\{c_1', c_1'', c_1''', 8\pi a^2\}$, then by (7.23) we have

$$F(\check{\zeta}_{c,\xi}) + \frac{1}{2}\gamma_{c,\xi} \geq \frac{1}{4\pi}\log\frac{c^{1/2}}{2a} - C_1.$$

Since $F(\check{\zeta}_{c,\xi}) \leq \beta(k)$ we finally obtain

$$\gamma_{c,\xi} \geq \frac{1}{2\pi}\log\frac{c^{1/2}}{2a} - (C_1 + 2\beta(k)).$$

This readily implies (7.18), for $c_1 := \max\{c_1', c_1'', c_1''', 8\pi a^2\}$ and $\xi_1 := \xi_1'$.$\diamondsuit$

7.5 Proofs of the theorems

Proof of Theorem 1 Let c_1 and ξ_1 be as in Lemma 7. Let c_2 be a positive constant such that for $c \geq c_2$,

$$\frac{1}{2\pi}\log\frac{c^{1/2}}{2a} - C_2(k) > 1. \tag{7.24}$$

Let us consider $c \geq c_0 = \max\{1, c_1, c_2\}$, $\xi \geq \xi_1$ and $\check{\zeta}_{c,\xi} \in \Sigma_{c,\xi} \neq \emptyset$, by [Chapter 5, Lemma 5]. From (7.17), (7.24) and Lemma 7 we obtain

$$\mathrm{supp}(\check{\zeta}_{c,\xi}) \subset \mathrm{supp}(K_c\check{\zeta}_{c,\xi} - \eta_c), \tag{7.25}$$

modulo a set of measure zero. Let us observe that for $x \in \Omega_c$ we have

$$K_c\check{\zeta}_{c,\xi}(x) = KC^{-1}(\check{\zeta}_{c,\xi})(c^{1/2}x).$$

Moreover, since $KC^{-1}(\check{\zeta}_{c,\xi})(c^{1/2}x) \leq K_+C^{-1}(\check{\zeta}_{c,\xi})(c^{1/2}x)$ we infer

$$K_c\check{\zeta}_{c,\xi}(x) \leq K_+C^{-1}(\check{\zeta}_{c,\xi})(c^{1/2}x),$$

for $x \in \Omega_c$. Also, if $x \in \Omega_c$ is such that $|x| > 1$, then by applying (7.11) we obtain

$$\eta_c(x) \geq \frac{1}{2}x_1x_2,$$

hence, for $x \in \Omega_c$ such that $|x| > 1$, we derive

$$\begin{aligned} K_c\check{\zeta}_{c,\xi}(x) - \eta_c(x) &= K\mathcal{C}^{-1}(\check{\zeta}_{c,\xi})(c^{1/2}x) - \eta_c(x) \\ &\leq K_+\mathcal{C}^{-1}(\check{\zeta}_{c,\xi})(c^{1/2}x) - 1/2x_1x_2. \end{aligned}$$

By [Chapter 5, Lemma 7] we have

$$K_+\mathcal{C}^{-1}(\check{\zeta}_{c,\xi})(c^{1/2}x) - 1/2x_1x_2 \leq 0,$$

provided $|x| \geq Ac^{1/2}$, where A is some universal positive constant. Let us now define $R(c) := \max\{1, \xi_1, Ac^{1/2}\}$. Then

$$\operatorname{supp}(\check{\zeta}_{c,\xi}) \subset B_{R(c)}. \tag{7.26}$$

Fixing c, clearly, (7.26) holds for every $\xi \geq \xi_1$. Hence $\check{\zeta}_{c,R(c)} \in \Sigma_c$. This concludes the existence part of the theorem.

We now proceed to derive (7.4). Henceforth we assume c is a fixed number such that $c \geq c_0$. Consider $\check{\zeta}_c \in \Sigma_c$. Hence $\hat{\zeta}_\lambda := \mathcal{C}^{-1}(\check{\zeta}_c) \in \Sigma_\lambda$, where $\lambda = 1/c$. It is clear that $\operatorname{supp}(\hat{\zeta}_\lambda) \subset B_{c^{1/2}R(c)}$, modulo a set of zero measure. Applying Lemma 7 and (7.24) we find

$$K_c\check{\zeta}_c(x) - \eta_c(x) \geq 1,$$

for almost every $x \in \operatorname{supp}(\check{\zeta}_c)$, or equivalently

$$K\mathcal{C}^{-1}(\check{\zeta}_c)(c^{1/2}x) - \frac{1}{c}\eta(c^{1/2}x) \geq 1,$$

for almost every $x \in \operatorname{supp}(\check{\zeta}_c)$, whence $K\hat{\zeta}_\lambda(x) - \lambda\eta(x) \geq 1$, for almost every $x \in \operatorname{supp}(\hat{\zeta}_\lambda)$. Let us now observe that $\limsup_{|x|\to\infty}(K\hat{\zeta}_\lambda(x) - \lambda\eta(x)) \leq 0$. Hence there exists $M_3 \geq c^{1/2}R(c)$ such that $K\hat{\zeta}_\lambda(x) - \lambda\eta(x)) \leq 1/2$, provided $|x| > M_3$. Since $\hat{\zeta}_\lambda$ is a global maximiser of Ψ_λ relative to $\mathcal{F}$ we deduce that, in particular, $\hat{\zeta}_\lambda$ maximises Ψ_λ relative to functions in $\mathcal{F}$ which vanish outside $\Omega_{M_3} := B_{M_3} \cap \Omega$. Therefore, by Burton's theory, there exists an increasing function ϕ such that

$$\hat{\zeta}_\lambda = \phi \circ (K\hat{\zeta}_\lambda - \lambda\eta),$$

for almost every $x \in \Omega_{M_3}$. We modify ϕ by $\hat{\phi}_\lambda$ which is defined as follows

$$\hat{\phi}_\lambda(s) := \begin{cases} \phi_\lambda(s), & s \geq 1 \\ 0, & s < 1. \end{cases}$$

Therefore we derive

$$\hat{\zeta}_\lambda = \hat{\phi}_\lambda \circ (K\hat{\zeta}_\lambda - \lambda\eta),$$

for almost every $x \in \Omega$. Since $\hat{\zeta}_\lambda = -\Delta K\hat{\zeta}_\lambda$, for almost everywhere $x \in \Omega$, we deduce (7.4). Note that $\lambda_0 := 1/c_0$.$\diamondsuit$

To prove Theorem 2 we need the following lemma, the proof of which relies on

Proposition Suppose $Q \geq 1$ and $c \geq 4a^2$. Let $v \in L^p(\Pi_+)$ be a non-negative function vanishing outside a set of measure πa^2 and define $v_c(x) = \mathcal{C}(v)(x)$. For $x \in \Pi_+$ define $\hat{B}(x) := B_{Qa/c^{1/2}}(x)$ and

$$\begin{aligned} I(x) &:= \int_{\hat{B}(x)} \log\left(\frac{2a|x-\overline{y}|}{c^{1/2}|x-y|}\right) v_c(y)dy, \\ \overline{I}(x) &:= \int_{\hat{B}(x)} \log\left(\frac{2a|x-\underline{y}|}{c^{1/2}|x-y|}\right) v_c(y)dy. \end{aligned}$$

Then there exist constants N_1, N_2, N_3, N_1', N_2' and N_3' such that

$$I(x) \leq \begin{cases} (N_1 + N_2|\log x_2|)\|v\|_p, & x_2 \geq a \\ N_3\|v\|_p, & 0 < x_2 \leq a, \end{cases} \tag{7.27}$$

and

$$\overline{I}(x) \leq \begin{cases} (N_1' + N_2'|\log x_1|)\|v\|_p, & x_1 \geq a \\ N_3'\|v\|_p, & 0 < x_1 \leq a. \end{cases} \tag{7.28}$$

Proof see [4].$\diamondsuit$

Lemma 8 *(support lemma) Let c_0 be as in the proof of Theorem 1. Then there exist $\check{c} > c_0$ and $d > 0$ such that if $c \geq \check{c}$, then*

$$diam(supp(\check{\zeta}_c)) \leq d, \tag{7.29}$$

for all $\check{\zeta}_c \in \hat{\Sigma}_c$.

Proof Let $\check{c} := \max\{c_0, 4a^2\}$. Henceforth $c \geq \check{c}$ is fixed. Let us consider $\check{\zeta}_c \in \hat{\Sigma}_c$ (by Theorem 1, $\check{\zeta}_c$ exists). Let us recall that

$$K_c\check{\zeta}_c(x) - \eta_c(x) \geq \frac{1}{2\pi}\log\frac{c^{1/2}}{2a} - C_2(k),$$

for almost every $x \in \text{supp}(\check{\zeta}_c)$. Since $K_c\check{\zeta}_c(x) \leq K_+\check{\zeta}_c(x)$, for almost every $x \in \Omega_c$, we obtain

$$\begin{aligned}\eta_c(x) - C_2(k) &\leq K_+\check{\zeta}_c(x) - \frac{1}{2\pi}\log\frac{c^{1/2}}{2a} \\ &= \frac{1}{2\pi}\int_{\Omega_c}\log\frac{2a|x-\overline{y}||x-\underline{y}|}{c^{1/2}|x-y||x-\overline{y}|}\,\check{\zeta}_c(y)dy,\end{aligned}$$

for almost every $x \in \operatorname{supp}(\check{\zeta}_c)$. Let us define $S := \{x \in \Pi_+ |\ \min\{x_1, x_2\} > a\}$ and for $x \in \Pi_+$ define

$$J(x) := \frac{1}{2\pi}\int_{\hat{B}(x)}\log\frac{2a|x-\overline{y}||x-\underline{y}|}{c^{1/2}|x-y||x-\overline{y}|}\,\check{\zeta}_c(y)dy,$$

where $Q \geq 1$. Observe that for $x,\ y \in \Pi_+$ we have $|x-\overline{y}|/|x-\overline{y}| < 1$ and $|x-\underline{y}|/|x-\overline{y}| < 1$. Hence, on the one hand, for $x \in \Pi_+ \setminus S$ we can apply (7.27) and (7.28) to obtain

$$J(x) \leq \max\{I(x),\, \overline{I}(x)\} \leq N_3''\|\zeta_0\|_p, \tag{7.30}$$

where $N'' := \max\{N_3,\ N_3'\}$; and on the other hand, for $x \in S$ we obtain

$$J(x) \leq I(x) \leq (N_1 + N_2|\log x_2|)\ \|\zeta_0\|_p. \tag{7.31}$$

Therefore for almost every $x \in \operatorname{supp}(\check{\zeta}_c)$ we have

$$\begin{aligned}\eta_c(x) - C_2(k) &\leq \frac{1}{2\pi}\int_{\Omega_c\setminus\hat{B}(x)}\log\left(\frac{2a|x-\overline{y}|}{c^{1/2}|x-y|}\right)\check{\zeta}_c(y)dy \\ &+ \begin{cases}(N_1 + N_2|\log x_2|)\|\zeta_0\|_p, & x \in S \\ N_3''\|\zeta_0\|_p, & x \in \Pi_+ \setminus S.\end{cases}\end{aligned} \tag{7.32}$$

Since $\operatorname{supp}(\check{\zeta}_c)$ is essentially contained in $B_{R(c)}$, where $R(c) := \max\{1, \xi_1, Ac^{1/2}\}$ is as in the proof of Theorem 1, we obtain

$$\frac{1}{2\pi}\int_{\Omega_c\setminus\hat{B}(x)}\log\left(\frac{2a|x-\overline{y}|}{c^{1/2}|x-y|}\right)\check{\zeta}_c(y)dy \leq \frac{1}{2\pi}\log\frac{4R(c)}{Q}\int_{\Omega_c\setminus\hat{B}(x)}\check{\zeta}_c(y)dy, \tag{7.33}$$

for every $x \in \Pi_+$. Therefore by (7.33) and rearranging the terms in (7.32) we infer that for almost every $x \in \operatorname{supp}(\check{\zeta}_c)$ we have

$$\begin{aligned}&\frac{1}{2\pi}\log\frac{Q}{4R(c)}\int_{\Omega_c\setminus\hat{B}(x)}\check{\zeta}_c(y)dy \\ \leq &\begin{cases}N_3''\|\zeta_0\|_p + C_2(k) - \eta_c(x), & x \in \Pi_+ \setminus S \\ (N_1 + N_2|\log x_2|)\|)\|\zeta_0\|_p + C_2(k) - \eta_c(x), & x \in S.\end{cases}\end{aligned}$$

From this we deduce the existence of a positive constant ν such that

$$\log \frac{Q}{4R(c)} \int_{\Omega_c \setminus \hat{B}(x)} \check{\zeta}_c(y)dy < \nu. \tag{7.34}$$

Let us now set $Q := 4R(c)e^{2\nu}$, then from (7.34) we obtain

$$\int_{\Omega_c \setminus \hat{B}(x)} \check{\zeta}_c(y)dy < 1/2, \tag{7.35}$$

for almost every $x \in \operatorname{supp}(\check{\zeta}_c)$. We now claim that

$$\operatorname{diam}(\operatorname{supp}(\check{\zeta}_c)) \leq 8aR(c)e^{2\nu}/c^{1/2}.$$

To seek a contradiction suppose this is not true, that is, $\operatorname{diam}(\operatorname{supp}(\check{\zeta}_c)) > 8aR(c)e^{2\nu}/c^{1/2}$. In this case, there exist x_1 and x_2 in $\operatorname{den}(\operatorname{supp}(\check{\zeta}_c))$ such that $|x_1 - x_2| > 8aR(c)e^{2\nu}/c^{1/2}$. Hence $B_{8aR(c)e^{2\nu}/c^{1/2}}(x_1) \cap B_{8aR(c)e^{2\nu}/c^{1/2}}(x_2) = \emptyset$. Thus, by (7.35), we obtain

$$1 < \int_{B_{8aR(c)e^{2\nu}/c^{1/2}}(x_1)} \check{\zeta}_c(y)dy + \int_{B_{8aR(c)e^{2\nu}/c^{1/2}}(x_2)} \check{\zeta}_c(y)dy \leq 1,$$

which is a contradiction. Finally, since $R(c) := \max\{1,\ \xi_1,\ Ac^{1/2}\}$, (7.29) follows. In fact, we can take $d := 8ae^{2\nu}(1 + \xi_1 + A)$.$\diamondsuit$

Proof of Theorem 2 The proof consists of three assertions.
Assertion 1 There exist $c_2 \geq \check{c}$, $\overline{R} > 0$ such that if $c \geq c_2$ and $\check{\zeta}_c \in \Sigma_c$, then

$$\operatorname{supp}(\check{\zeta}_c) \subset L(2d) \cup (\Pi_+ \cap B_{\overline{R}}), \tag{7.36}$$

modulo a set of measure zero. Here

$$L(2d) := \{x \in \Pi_+ |\ \min\{x_1, x_2\} \leq 2d\}.$$

Proof of Assertion 1 To seek a contradiction, assume the assertion is false. Hence there exist sequences $\{c_j\}_{j=1}^{\infty}$, $\{\check{\zeta}_{c_j}\}_{j=1}^{\infty}$ and $\{x_j\}_{j=1}^{\infty}$ such that
(a) $\check{c} \leq c_j \to \infty$, as $j \to \infty$.
(b) $\check{\zeta}_{c_j} \in \Sigma_{c_j}$, for all j.
(c) $x_j = (x_{j,1}, x_{j,2}) \in \operatorname{den}(\operatorname{supp}(\check{\zeta}_{c_j}))$, for all j, and $|x_j| \to \infty$, as $j \to \infty$.

For simplicity, when appropriate, we replace "c_j" by "j", e.g. we write $\hat{\Psi}_j$ for $\hat{\Psi}_{c_j}$, $\check{\zeta}_j$ for $\check{\zeta}_{c_j}$, etc. The Schwarz-symmetrisation of $\check{\zeta}_j$ with respect to x_0 is denoted by

ζ_j^*. Without loss of generality we may assume that $\text{supp}(\zeta_j^*) \subset B_{1/4\sqrt{2\pi}}(x_0)$ for all j. Fixing j for now, we observe that from Lemma 8, $\text{supp}(\check{\zeta}_j)$ is essentially contained in $\Pi_+ \setminus L(d)$. Therefore

$$\begin{aligned}
&\hat{\Psi}_j(\zeta_j^*) - \hat{\Psi}_j(\check{\zeta}_j) \\
&\geq \int_{\Omega_j}\int_{\Omega_j} \left(\frac{1}{2}h(c_j^{1/2}x, c_j^{1/2}y) - \frac{1}{4\pi}\log\frac{1}{c_j^{1/2}} + \eta_j(x)\right) \check{\zeta}_j(x)\check{\zeta}_j(y)dxdy \\
&- \int_{\Omega_j}\int_{\Omega_j} \left(\frac{1}{2}h(c_j^{1/2}x, c_j^{1/2}y) - \frac{1}{4\pi}\log\frac{1}{c_j^{1/2}} + \eta_j(x)\right) \zeta_j^*(x)\zeta_j^*(y)dxdy, \quad (7.37)
\end{aligned}$$

where we have applied an n-dimensional generalisation [7] of an inequality of F. Riesz on rearrangements to the logarithmic parts. Since $h \geq h_+$ we infer

$$\begin{aligned}
&\frac{1}{2}h(c_j^{1/2}x, c_j^{1/2}y) - \frac{1}{4\pi}\log\frac{1}{c_j^{1/2}} + \eta_j(x) \\
&\geq \frac{1}{4\pi}\log\frac{|x-\overline{y}|}{c_j^{1/2}|x-\overline{y}||x-\underline{y}|} - \frac{1}{4\pi}\log\frac{1}{c_j^{1/2}} + x_1x_2 - \frac{x_1x_2}{c_j^2|x|^4} \\
&\geq \frac{1}{4\pi}\log\frac{|x-\underline{y}|}{|x-\overline{y}||x-\underline{y}|} + \frac{1}{2}x_1x_2, \qquad (7.38)
\end{aligned}$$

for all $x,\ y \in \text{den}(\text{supp}(\check{\zeta}_j))$. Also, note that for $x,\ y \in \text{supp}(\zeta_j^*) \subset B_{1/4\sqrt{2\pi}}(x_0)$ we have

$$\begin{aligned}
&\frac{1}{2}h(c_j^{1/2}x, c_j^{1/2}y) - \frac{1}{4\pi}\log\frac{1}{c_j^{1/2}} + \eta_j(x) \\
&\leq \frac{1}{2}\hat{h}(c_j^{1/2}x, c_j^{1/2}y) + \frac{1}{2}h_+(c_j^{1/2}x, c_j^{1/2}y) - \frac{1}{4\pi}\log\frac{1}{c^{1/2}} + x_1x_2 \\
&\leq \frac{1}{2}\hat{h}_1(c_j^{1/2}x, c_j^{1/2}y) + \frac{1}{4\pi}\log\frac{|x-\overline{y}|}{|x-\overline{y}||x-\underline{y}|} + x_1x_2 \\
&\leq \frac{c_jx_1x_2}{2\pi(c_j|x||y|-1)^2} + \frac{1}{4\pi}\log\frac{|x-\overline{y}|}{|x-\overline{y}||x-\underline{y}|} + x_1x_2 \leq K_1, \qquad (7.39)
\end{aligned}$$

where K_1 is a positive constant, provided j is sufficiently large. Let us assume that $x_{j,2} \to \infty$, as $j \to \infty$ (the case where $x_{j,1} \to \infty$ can be treated similarly); moreover we may assume that for almost every $x \in \text{supp}(\check{\zeta}_j)$ we have $x_2 \leq x_{j,2}$. Therefore, by (7.38), for $x,\ y \in \text{den}(\text{supp}(\check{\zeta}_j))$ we find

$$\frac{1}{2}h(c_j^{1/2}x, c_j^{1/2}y) - \frac{1}{4\pi}\log\frac{1}{c_j^{1/2}} + \eta_j(x) \quad \geq \quad \frac{1}{4\pi}\log\frac{1}{|x-\overline{y}|} + \frac{1}{2}d(x_{j,2} - d)$$

$$\begin{aligned} &\geq \frac{1}{4\pi}\log\frac{1}{2x_{j,2}} + \frac{1}{2}d(x_{j,2} - d) \\ &\to \infty, \end{aligned} \tag{7.40}$$

as $j \to \infty$. Hence from (7.39), (7.40) and (7.37) we deduce

$$\hat{\Psi}_{j_0}(\check{\zeta}_{j_0}) - \hat{\Psi}_{j_0}(\zeta^*_{j_0}) > 0,$$

for some j_0 sufficiently large. This contradicts the maximality of $\check{\zeta}_{j_0}$. Hence the assertion is proved.

Assertion 2 There exists $\mu(k) > 0$ such that if $c \geq c_2$ and $\check{\zeta}_c \in \Sigma_c$, then

$$\mathrm{supp}(\check{\zeta}_c) \subset \{x \in \Pi_+ |\ x_1 x_2 < \mu(k)\}, \tag{7.41}$$

modulo a set of measure zero.

Proof of Assertion 2 Let us fix $c \geq c_2$ and consider $\check{\zeta}_c \in \Sigma_c$. From Assertion 1 it follows that there exists $\tau > a$ such that $\mathrm{supp}(\check{\zeta}_c)$ is essentially contained in $\mathbb{R}^+ \times (0,\tau)$ or $(0,\tau) \times \mathbb{R}^+$. Let us first suppose that $\mathrm{supp}(\check{\zeta}_c)$ is essentially contained in $\mathbb{R}^+ \times (0,\tau)$. Then for almost every $x \in \mathrm{supp}(\check{\zeta}_c)$, such that $x_1 > \tau$, we have (see 7.17 and 7.18)

$$K_+\check{\zeta}_c(x) - \frac{1}{2}x_1x_2 \geq K_c\check{\zeta}_c(x) - \eta_c(x) \geq \frac{1}{2\pi}\log\frac{c^{1/2}}{2a} - C_2(k).$$

Therefore we obtain

$$\begin{aligned} x_1x_2 &\leq 2K_+\check{\zeta}_c(x) - \frac{1}{\pi}\log\frac{c^{1/2}}{2a} + 2C_2(k) \\ &= \frac{1}{\pi}\int_{\Omega_c}\log\frac{2a|x-\overline{y}||x-\underline{y}|}{c^{1/2}|x-y||x-\underline{\overline{y}}|}\,\check{\zeta}_c(y)dy + 2C_2(k). \end{aligned} \tag{7.42}$$

The integral in (7.42) is dominated by $\int_{\Omega_c}\log\frac{2a|x-\overline{y}|}{c^{1/2}|x-y|}\,\check{\zeta}_c(y)dy$. Now we write

$$\begin{aligned} \int_{\Omega_c}\log\frac{2a|x-\overline{y}|}{c^{1/2}|x-y|}\,\check{\zeta}_c(y)dy &= \int_{B_{a/c^{1/2}}(x)}\log\frac{2a|x-\overline{y}|}{c^{1/2}|x-y|}\,\check{\zeta}_c(y)dy \\ &+ \int_{\Omega_c\setminus B_{a/c^{1/2}}(x)}\log\frac{2a|x-\overline{y}|}{c^{1/2}|x-y|}\,\check{\zeta}_c(y)dy. \end{aligned}$$

Applying (7.27), with $Q = 1$, we find $\overline{\eta} > 0$ such that

$$\int_{B_{a/c^{1/2}}(x)}\log\frac{2a|x-\overline{y}|}{c^{1/2}|x-y|}\,\check{\zeta}_c(y)dy \leq \overline{\eta}. \tag{7.43}$$

We also have
$$\int_{\Omega_c \setminus B_{a/c^{1/2}}(x)} \log \frac{2a|x-\overline{y}|}{c^{1/2}|x-y|}\, \check{\zeta}_c(y)dy \leq \log(4\tau). \tag{7.44}$$

Hence from (7.43) and (7.44) we conclude that for almost every $x \in \text{supp}(\check{\zeta}_c)$ we have

$$x_1 x_2 \leq \frac{1}{\pi}(\overline{\eta} + \log(a\tau)) + 2C_2(k).$$

The case where $\text{supp}(\check{\zeta}_c)$ is essentially contained in $(0,\tau) \times \mathbb{R}^+$ can be treated similarly (in the proof we make use of (7.27)). Therefore we derive the desired result.

Assertion 3 There exist $c_3 \geq c_2$, $R > 0$ such that if $c \geq c_3$ and $\check{\zeta}_c \in \Sigma_c$, then

$$\text{supp}(\check{\zeta}_c) \subset B_R,$$

modulo a set of measure zero.

Proof of Assertion 3 To seek a contradiction, suppose the assertion is false. Then by Assertion 2, there exist sequences $\{c_j\}_{j=1}^{\infty}$, $\{\check{\zeta}_j\}_{j=1}^{\infty}$ and $\{x_j\}_{j=1}^{\infty}$ such that the following hold

(a') $c_2 \leq c_j \to \infty$, as $j \to \infty$.

(b') $\check{\zeta}_j \in \Sigma_j$ such that $\text{supp}(\check{\zeta}_j)$ is essentially contained in $\mathbb{R}^+ \times (0, 1/j)$, for all j; the case where $\text{supp}(\check{\zeta}_j)$ is essentially contained in $(0, 1/j) \times \mathbb{R}^+$ can be treated similarly.

(c') $x_j \in \text{den}(\text{supp}(\check{\zeta}_j))$, for every j, and $|x_j| \to \infty$ as $j \to \infty$.

Notice that we have again adopted the convention of replacing "c_j" by "j", when appropriate.

We claim that $\text{diam}(\text{supp}(\check{\zeta}_j)) \to 0$, as $j \to \infty$. To prove the claim we fix j. Then for almost every $x \in \text{supp}(\check{\zeta}_j)$ we have

$$\begin{aligned}
\frac{1}{2\pi}\log\frac{c_j^{1/2}}{2a} - C_2(k) &\leq K_j\check{\zeta}_j(x) - \eta_j(x) \\
&\leq K_+\check{\zeta}_j(x) \leq \frac{1}{2\pi}\int_{\Omega_j} \log\frac{|x-\overline{y}|}{|x-y|}\, \check{\zeta}_j(y)dy,
\end{aligned}$$

since η_j is positive and $K_j\check{\zeta}_j(x) \leq K_+\check{\zeta}_j(x)$ for almost every $x \in \Omega_j$. Therefore,

$$-C_2(k) \leq \frac{1}{2\pi}\int_{\Omega_j} \log\frac{2a|x-\overline{y}|}{c_j^{1/2}|x-y|}\, \check{\zeta}_j(y)dy.$$

Let $Q > 1$ and set $\hat{B}(x) := B_{Qa/c_j^{1/2}}(x)$. Then

$$-C_2(k) \leq \frac{1}{2\pi}\int_{\hat{B}(x)} \log \frac{2a|x-\overline{y}|}{c_j^{1/2}|x-y|}\, \check{\zeta}_j(y)dy + \frac{1}{2\pi}\int_{\Omega_j\setminus\hat{B}(x)} \log \frac{2a|x-\overline{y}|}{c_j^{1/2}|x-y|}\, \check{\zeta}_j(y)dy.$$

Applying (7.27) we find that

$$\frac{1}{2\pi}\int_{\hat{B}(x)} \log \frac{2a|x-\overline{y}|}{c_j^{1/2}|x-y|}\, \check{\zeta}_j(y)dy \leq K_2.$$

Also we have

$$\frac{1}{2\pi}\int_{\Omega_j\setminus\hat{B}(x)} \log \frac{2a|x-\overline{y}|}{c_j^{1/2}|x-y|}\, \check{\zeta}_j(y)dy \leq \frac{1}{2\pi}\log\frac{2(d+2)}{Q}\int_{\Omega_j\setminus\hat{B}(x)} \check{\zeta}_j(y)dy,$$

where we have used the fact that for $x,\ y \in \operatorname{supp}(\check{\zeta}_j)$,

$$\begin{aligned} |x-\overline{y}| \leq |x-y| + |y-\overline{y}| &\leq \operatorname{diam}(\operatorname{supp}(\check{\zeta}_j)) + 2y_2 \\ &\leq \operatorname{diam}(\operatorname{supp}(\check{\zeta}_j)) + 2/j, \end{aligned} \tag{7.45}$$

and that $\operatorname{diam}(\operatorname{supp}(\check{\zeta}_j)) + 2/j \leq d+2$. Therefore, we obtain

$$-C_2(k) \leq K_2 + \frac{1}{2\pi}\log\frac{2(d+2)}{Q}\int_{\Omega_j\setminus\hat{B}(x)} \check{\zeta}_j(y)dy,$$

hence by rearranging we derive

$$\log\frac{Q}{2(d+2)}\int_{\Omega_j\setminus\hat{B}(x)} \check{\zeta}_j(y)dy \leq 2\pi(C_2(k)+K_2) < \hat{\nu},$$

for some $\hat{\nu} > 0$. Let us now set $Q := 2(d+2)e^{2\hat{\nu}}$ to obtain

$$\int_{\Omega_j\setminus\hat{B}(x)} \check{\zeta}_j(y)dy < \frac{1}{2}, \tag{7.46}$$

for almost every $x \in \operatorname{supp}(\check{\zeta}_j)$. From (7.46), applying the same argument as in Lemma 8, we find

$$\operatorname{diam}(\operatorname{supp}(\check{\zeta}_j)) < \frac{4a(d+2)e^{2\hat{\nu}}}{c_j^{1/2}}.$$

Letting $j \to \infty$ we obtain the desired result.

Let ζ_j^* denote the Schwarz-symmetrisation of $\check{\zeta}_j$ with respect to x_0. To derive a contradiction it suffices to show $\hat{\Psi}_{j_1}(\zeta_{j_1}^*) - \hat{\Psi}_{j_1}(\check{\zeta}_{j_1}) > 0$, for some j_1. However, this is easily proved using the same method employed in Assersion 1; the key step being a result similar to (7.40) which is explained here. For $x,\ y \in \mathrm{den}(\mathrm{supp}(\check{\zeta}_j))$ we have

$$
\begin{aligned}
\frac{1}{2}h(c_j^{1/2}x, c_j^{1/2}y) - \frac{1}{4\pi}\log\frac{1}{c_j^{1/2}} + \eta_j(x) &\geq \frac{1}{2}h_+(c_j^{1/2}x, c_j^{1/2}y) - \frac{1}{4\pi}\log\frac{1}{c_j^{1/2}} \\
&= \frac{1}{4\pi}\log\frac{|x-\overline{y}|}{|x-\overline{y}||x-\underline{y}|} \\
&\geq \frac{1}{4\pi}\log\frac{1}{|x-\overline{y}|};
\end{aligned}
\tag{7.47}
$$

now by using (7.45) we obtain

$$
\begin{aligned}
\frac{1}{4\pi}\log\frac{1}{|x-\overline{y}|} &\geq \frac{1}{4\pi}\log\frac{1}{\mathrm{diam}(\mathrm{supp}(\check{\zeta}_j)) + 2/j} \\
&\to 0,
\end{aligned}
\tag{7.48}
$$

as $j \to \infty$. This completes the proof of the assersion, hence the theorem.$\diamondsuit$

Proof of Theorem 3 Let us first point out that by Theorem 2, sequences like $\{\hat{x}_j\}_{j=1}^{\infty}$ do exist; indeed, if we consider sequences $\{c_j\}_{j=1}^{\infty}$, $\{\check{\zeta}_j\}_{j=1}^{\infty}$ satisfying (i) and (ii), then, by Theorem 2, $\{\hat{x}_j\}_{j=1}^{\infty}$ is bounded, hence it contains a convergent subsequence. By Theorem 2 we can assume, without loss of generality, $\hat{x}_j \to \hat{x}$ as $j \to \infty$. We then need to show $\hat{x} = x_0$.

By maximality of $\check{\zeta}_j$ we have

$$
\hat{\Psi}_j(\check{\zeta}_j) \geq \hat{\Psi}_j(\zeta_j^*),
\tag{7.49}
$$

where ζ_j^* denotes the Schwarz-symmetrisation of $\check{\zeta}_j$ with respect to x_0. From (7.49) and an application of Riesz's inequality we obtain

$$
\begin{aligned}
T_j(\check{\zeta}_j) &:= \int_{\Omega_j}\int_{\Omega_j}\left(\frac{1}{2}h(c_j^{1/2}x, c_j^{1/2}y) - \frac{1}{4\pi}\log\frac{1}{c_j^{1/2}} + \eta_j(x)\right)\check{\zeta}_j(x)\check{\zeta}_j(y)dxdy \\
&\leq \int_{\Omega_j}\int_{\Omega_j}\left(\frac{1}{2}h(c_j^{1/2}x, c_j^{1/2}y) - \frac{1}{4\pi}\log\frac{1}{c_j^{1/2}} + \eta_j(x)\right)\zeta_j^*(x)\zeta_j^*(y)dxdy \\
&= T_j(\zeta_j^*).
\end{aligned}
\tag{7.50}
$$

We now fix $0 < \epsilon < 1/(2\sqrt{2\pi})$. Then there exists $j_0(\epsilon) \in \mathbb{N}$ such that if $j \geq j_0(\epsilon)$, then

$\mathrm{supp}(\zeta_j^*) \subset B_\epsilon(x_0)$. For $j \geq j_0(\epsilon)$ and $x,\ y \in B_\epsilon(x_0)$ we have

$$\begin{aligned}
&\left|\frac{1}{2}h(c_j^{1/2}x, c_j^{1/2}y) - \frac{1}{4\pi}\log\frac{1}{c_j^{1/2}} - H_1(x_0)\right| \\
\leq\ & \frac{1}{2}\hat{h}_1(c_j^{1/2}x, c_j^{1/2}y) + \frac{1}{4\pi}\left|\log\frac{2x_{0,1}x_{0,2}|x-\overline{y}|}{|x-\overline{y}||x-\underline{y}||x_0|}\right| \\
\leq\ & \frac{c_j x_2 y_2}{2\pi(c_j|x||y|-1)^2} + \frac{1}{4\pi}\left|\log\frac{2x_{0,1}x_{0,2}|x-\overline{y}|}{|x-\overline{y}||x-\underline{y}||x_0|}\right| \\
\leq\ & \frac{c_j(x_{0,1}+\epsilon)(x_{0,2}+\epsilon)}{2\pi(c_j(|x_0|-\epsilon)^2-1)^2} + \frac{1}{4\pi}\sup_{x,y\in B_\epsilon(x_0)}\left|\log\frac{2x_{0,1}x_{0,2}|x-\overline{y}|}{|x-\overline{y}||x-\underline{y}||x_0|}\right| \\
\to\ & 0,
\end{aligned}$$

as $j \to \infty$. This implies

$$\int_{\Omega_j}\int_{\Omega_j}\left(\frac{1}{2}h(c_j^{1/2}x, c_j^{1/2}y) - \frac{1}{4\pi}\log\frac{1}{c_j^{1/2}}\right)\zeta_j^*(x)\zeta_j^*(y)dxdy \to H_1(x_0), \tag{7.51}$$

as $j \to \infty$. Also, from (7.11), we have

$$|\eta_j(x) - H_2(x_0)| \leq |x_1x_2 - x_{0,1}x_{0,2}| + \frac{1}{c_j^2|x|^2},$$

for $x \in B_\epsilon(x_0)$, hence

$$\sup_{x\in B_\epsilon(x_0)}|\eta_j(x) - H_0(x)| \to 0, \tag{7.52}$$

as $j \to \infty$. From (7.51) and (7.52) we deduce that $T_j(\zeta_j^*) \to H(x_0)$, as $j \to \infty$.

We now claim that

$$T_j(\check{\zeta}_j) \to H(\hat{x}),\ \text{ as } j \to \infty. \tag{7.53}$$

Note that by proving (7.53) we will have completed the proof of the theorem; indeed, if (7.53) is true, then from (7.50) we infer $H(\hat{x}) \leq H(x_0)$, hence $\hat{x} = x_0$. To prove the claim we first show that $\hat{x} \notin \partial\Pi_+$. Seeking a contradiction we suppose $\hat{x} \in \partial\Pi_+$. Setting $\hat{x} := (\hat{x}_1, \hat{x}_2)$, we may assume that $\hat{x}_2 = 0$. Fix $\epsilon > 0$; then there exists $\overline{j}(\epsilon) \in \mathbb{N}$ such that if $j \geq \overline{j}(\epsilon)$, then

$$|x - \hat{x}| \leq |x - \hat{x}_j| + |\hat{x}_j - \hat{x}| \leq \mathrm{diam}(\mathrm{supp}(\check{\zeta}_j)) + \epsilon, \tag{7.54}$$

for almost every $x \in \operatorname{supp}(\check{\zeta}_j)$. Now, similarly to (7.47) and (7.48) we derive

$$\begin{aligned}\frac{1}{2}h(c_j^{1/2}x, c_j^{1/2}y) - \frac{1}{4\pi}\log\frac{1}{c_j^{1/2}} + \eta_j(x) &\geq \frac{1}{4\pi}\log\frac{1}{|x-\overline{y}|} \\ &\geq \frac{1}{4\pi}\log\frac{1}{\operatorname{diam}(\operatorname{supp}(\check{\zeta}_j)) + 2y_2},\end{aligned}$$

for $x,\ y \in \operatorname{den}(\operatorname{supp}(\check{\zeta}_j))$ and $j \geq \overline{j}(\epsilon)$. If $j \geq \overline{j}(\epsilon)$ and $y \in \operatorname{supp}(\check{\zeta}_j)$, we can apply (7.54) to derive

$$y_2 \leq |y - \hat{x}| \leq \operatorname{diam}(\operatorname{supp}(\check{\zeta}_j)) + \epsilon.$$

Therefore if $j \geq \overline{j}(\epsilon)$ and $x,\ y \in \operatorname{den}(\operatorname{supp}(\check{\zeta}_j))$, then

$$\frac{1}{2}h(c_j^{1/2}x, c_j^{1/2}y) - \frac{1}{4\pi}\log\frac{1}{c_j^{1/2}} + \eta_j(x) \geq \frac{1}{4\pi}\log\frac{1}{3\operatorname{diam}(\operatorname{supp}(\check{\zeta}_j)) + 2\epsilon}.$$

This in conjunction with the fact that $\operatorname{diam}(\operatorname{supp}(\check{\zeta}_j)) \to 0$, as $j \to \infty$, implies

$$\limsup_{j\to\infty} T_j(\check{\zeta}_j) \geq \frac{1}{4\pi}\log\frac{1}{2\epsilon},$$

hence $\limsup_{j\to\infty} T_j(\check{\zeta}_j) = \infty$, since $\epsilon > 0$ was arbitrary. However, this contradicts (7.50), since $T_j(\zeta_j^*)$ is bounded from above for all sufficiently large j, see (7.39). Hence, $\hat{x} \notin \partial\Pi_+$.

Let $\varepsilon := \operatorname{dist}(\hat{x}, \partial\Pi_+)$, the distance from $\hat{x}$ to $\partial\Pi_+$. As shown above $\varepsilon > 0$. Observe that there exists $\check{j}(\varepsilon) \in \mathbb{N}$ such that if $j \geq \check{j}(\varepsilon)$, then $\operatorname{supp}(\check{\zeta}_j)$ is essentially contained in $B_{\varepsilon/2}(\hat{x})$. We now define the real valued function D on $\Omega \times \Omega \subset \mathbb{R}^4$ as

$$D(x,y) := D(x_1, x_2, y_1, y_2) = h_+(x,y).$$

Note that ∇D is bounded on $B_{\varepsilon/2}(\hat{x}) \times B_{\varepsilon/2}(\hat{x})$, that is, there exists a positive constant, say K_3, such that $\|\nabla D\|_{\infty, B_{\varepsilon/2}(\hat{x})\times B_{\varepsilon/2}(\hat{x})} \leq K_3$. Now, for $j \geq \check{j}(\varepsilon)$ and $x,\ y \in B_{\varepsilon/2}(\hat{x})$ we can apply Mean Value Inequality to deduce

$$\begin{aligned}&\left|\frac{1}{2}h(c_j^{1/2}x, c_j^{1/2}y) - \frac{1}{4\pi}\log\frac{1}{c_j^{1/2}} - H_1(\hat{x})\right| \\ \leq\ & \frac{1}{2}\hat{h}_1(c_j^{1/2}x, c_j^{1/2}y) + \frac{1}{4\pi}|D(x,y) - D(\hat{x},\hat{x})| \\ \leq\ & \frac{1}{2}\hat{h}_1(c_j^{1/2}x, c_j^{1/2}y) + \frac{1}{4\pi}K_3|(x,y) - (\hat{x},\hat{x})|_{\mathbb{R}^4} \\ =\ & o(1) + \frac{1}{4\pi}K_3|(x,y) - (\hat{x},\hat{x})|_{\mathbb{R}^4}, \qquad (7.55)\end{aligned}$$

as $j \to \infty$, uniformly in x and y, where $|\cdot|_{\mathbb{R}^4}$ denotes the usual Euclidean norm in $\mathbb{R}^4$. Note that for $j \geq \check{j}(\varepsilon)$ and $x,\ y \in B_{\varepsilon/2}(\hat{x})$ we have

$$\begin{aligned} |(x,y)-(\hat{x},\hat{x})|_{\mathbb{R}^4} &\leq 4(\mathrm{diam}(\mathrm{supp}(\check{\zeta}_j)) + |\hat{x}_j - \hat{x}|) \\ &= o(1), \end{aligned} \tag{7.56}$$

as $j \to \infty$. Whence by (7.55) and (7.56) we obtain

$$\frac{1}{2}h(c_j^{1/2}x, c_j^{1/2}y) - \frac{1}{4\pi}\log\frac{1}{c_j^{1/2}} - H_1(\hat{x}) = o(1),$$

as $j \to \infty$, uniformly in $x,\ y \in B_{\varepsilon/2}(\hat{x})$. This, in turn, shows that

$$\int_{\Omega_j}\int_{\Omega_j}\left(\frac{1}{2}h(c_j^{1/2}x, c_j^{1/2}y) - \frac{1}{4\pi}\log\frac{1}{c_j^{1/2}}\right)\check{\zeta}_j(x)\check{\zeta}_j(y)dxdy \to H_1(\hat{x}),$$

as $j \to \infty$. To complete the proof it remains to show that

$$\eta_j(x) - H_0(\hat{x}) = o(1), \tag{7.57}$$

as $j \to \infty$, uniformly in $x \in B_{\varepsilon/2}(\hat{x})$. So let us fix $x \in B_{\varepsilon/2}(\hat{x})$ and suppose $j \geq \check{j}(\varepsilon)$. Then

$$|\eta_j(x) - H_0(\hat{x})| \leq |x_1x_2 - \hat{x}_1\hat{x}_2| + \frac{1}{c_j^2|x|^2}.$$

Setting $\hat{x}_j := (\hat{x}_{j,1}, x_{j,2})$ and $\hat{x} := (\hat{x}_1, \hat{x}_2)$, it is readily verified that

$$\begin{aligned} |x_1x_2 - \hat{x}_1\hat{x}_2| \leq |x_2|\mathrm{diam}(\mathrm{supp}(\check{\zeta}_j)) &+ |\hat{x}_{j,1}|\mathrm{diam}(\mathrm{supp}(\check{\zeta}_j)) \\ &+ |\hat{x}_{j,1}\hat{x}_{j,2} - \hat{x}_1\hat{x}_2| \to 0, \end{aligned}$$

uniformly in x. From this (7.57) clearly follows, hence

$$\int_{\Omega_j}\eta_j(x)\check{\zeta}_j(x)dx \to H_0(\hat{x}),$$

as $j \to \infty$. So we are done.$\diamondsuit$

7.6 Solutions of $P(\lambda)$ as weak solutions of Euler equations

Let U and P denote the velocity field and the pressure of an ideal fluid of unit density in Ω. Then U and P satisfy the Euler equations

$$(U \cdot \nabla)U = -\nabla P \text{ in } \Omega \tag{7.58}$$

$$\nabla \cdot U = 0 \text{ in } \Omega \tag{7.59}$$

$$U \cdot \vec{n} = 0 \text{ on } \partial\Omega. \tag{7.60}$$

In [44] a weak formulation of (6.1)-(6.3) is derived in terms of the vorticity function ω and the stream funtion ψ, namely

$$\int_\Omega \omega \, [\psi, \phi] = 0, \tag{7.61}$$

for all $\phi \in C_0^\infty(\Omega)$, where $[\cdot,\cdot]$ stands for the Jacobian. Writing (7.61) in the context of our work in the present chapter we obtain

$$H(\zeta, \phi) := \int_\Omega \zeta \, [K\zeta - \lambda\eta, \phi] = 0, \tag{7.62}$$

for all $\phi \in C_0^\infty(\Omega)$. Equation (7.62) is called the weak vorticity-formulation of the Euler equations. We now present a heuristic argument for believing $H(\hat{\zeta}, \phi) = 0$, for all $\phi \in C_0^\infty(\Omega)$: Since $\hat{\zeta}$ has compact support there exists Ω_N such that $\operatorname{supp}(\hat{\zeta}) \subset \Omega_N$. Hence it suffices to show

$$H(\hat{\zeta}, \phi) = 0, \tag{7.63}$$

for all $\phi \in C_0^\infty(\Omega_N)$. We fix $\phi \in C_0^\infty(\Omega_N)$ and denote by $\xi_t(x)$ the unique solution of the Hamiltonian system

$$\frac{dz}{dt} = \nabla^\perp \phi(z),$$

satisfying the initial condition $z(0) = x \in \Omega_N$. It is a standard result that the mapping $x \to \xi_t(x)$, $t \in [-T, T]$, T small, defines a one-parameter family of measure preserving diffeomorphisms of Ω_N, see for example [36]. Now following [44] we obtain

$$\Psi(\hat{\zeta} \circ \xi_t^{-1}) = \Psi(\hat{\zeta}) + t \int_{\Omega_N} \hat{\zeta} \, [K\hat{\zeta}, \phi] + o(t),$$

as $t \to 0$. Futhermore, observe that

$$\lim_{t \to 0} \frac{\eta \circ \xi_t(x) - \eta(x)}{t} = \nabla\eta(x) \cdot \nabla^\perp \phi(x) = [\eta, \phi](x).$$

Hence by applying the Lebesgue Dominated Convergence Theorem we obtain

$$\Im(\hat{\zeta} \circ \xi_t^{-1}) = \Im(\hat{\zeta}) + t \int_{\Omega_N} \hat{\zeta}\ [\eta, \phi] + o(t),$$

as $t \to 0$. Therefore

$$\Psi_\lambda(\hat{\zeta} \circ \xi_t^{-1}) = \Psi_\lambda(\hat{\zeta}) + t \int_{\Omega_N} \hat{\zeta}\ [K\hat{\zeta} - \lambda\eta, \phi] + o(t),$$

as $t \to 0$, since $[K\hat{\zeta}, \phi] - \lambda[\eta, \phi] = [K\hat{\zeta} - \lambda\eta, \phi]$. Therefore, if $\kappa(t) := \Psi_\lambda(\hat{\zeta} \circ \xi_t^{-1})$, we have proved

$$\kappa'(0) = H(\hat{\zeta}, \phi).$$

Since $\hat{\zeta} \in \Sigma(\lambda)$ and $\hat{\zeta} \circ \xi_t^{-1} \in \mathcal{F}$, κ has a global maximiser at 0, in $[-T, T]$, whence $\kappa'(0) = 0$ and we obtain (7.63) as desired.

Appendix

Let $p > 2$ and D be a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary ∂D. Let $T : L^p(D) \to H_0^1(D)$ be the map such that for $\zeta \in L^p(D)$, $T\zeta$ denotes the unique minimiser of

$$F(u) := \frac{1}{2}\int_D |\nabla u|^2 - \int_D u\zeta,$$

relative to $u \in H_0^1(D)$. Then

$$T\zeta(x) = \int_D G(x,y)\,\zeta(y)dy, \text{ a.e. in } D,\ \zeta \in L^p(D),$$

where G denotes the Green's function for $-\Delta$ on D with homogeneous Dirichlet boundary conditions.

Proof Let us first consider $0 \le \zeta \in L^p(D)$ and define

$$\zeta_n(x) := \begin{cases} n, & \zeta(x) \ge n \\ \zeta(x), & \zeta(x) < n, \end{cases}$$

for $n \in \mathbb{N}$. It is a standard result that $T : L^p(D) \to H_0^1(D)$ is a bounded linear operator. Therefore, since $\zeta_n \to \zeta$ in $L^p(D)$, as $n \to \infty$, we deduce $T\zeta_n \to T\zeta$ in $H^1(D)$, as $n \to \infty$. Since $H^1(D)$ is continuously embedded into $L^2(D)$ we infer that $T\zeta_n \to T\zeta$ in $L^2(D)$ as $n \to \infty$. This, in turn, implies existence of a subsequence, say $\{T\zeta_{n_j}\}_{j=1}^{\infty}$ such that

$$T\zeta_{n_j}(x) \to T\zeta(x),$$

for almost every $x \in D$, as $j \to \infty$. An application of [17, Chapter 2, Proposition 24] implies

$$T\zeta_{n_j}(x) = \int_D G(x,y)\,\zeta_{n_j}(y)dy, \text{ a.e. in } D,$$

for every j. Since $\zeta_{n_j}(x) \to \zeta(x)$, for every $x \in D$, as $j \to \infty$ we can apply the Monotone Convergence Theorem to derive

$$\int_D G(x,y)\,\zeta_{n_j}(y)dy \to \int_D G(x,y)\,\zeta(y)dy,$$

for every $x \in D$, as $j \to \infty$. Therefore

$$T\zeta_{n_j}(x) \to \int_D G(x,y)\,\zeta(y)dy,$$

for almost every $x \in D$, as $j \to \infty$. Hence we conclude

$$T\zeta(x) = \int_D G(x,y)\,\zeta(y)dy,$$

for almost every $x \in D$.

For arbitrary $\zeta \in L^p(D)$, we write $\zeta = \zeta^+ - \zeta^-$. Since T is linear we have $T\zeta = T\zeta^+ - T\zeta^-$. Now using the above argument for each ζ^+ and ζ^- we obtain

$$\begin{aligned} T\zeta(x) &= \int_D G(x,y)\,\zeta^+(y)dy - \int_D G(x,y)\,\zeta^-(y)dy \\ &= \int_D G(x,y)(\zeta^+(y) - \zeta^-(y))dy = \int_D G(x,y)\,\zeta(y)dy, \end{aligned}$$

for almost every $x \in D$. Hence we are done.$\diamondsuit$

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